VOL. 24, No. 2, NOV. - DEC., 1950.

# MATHEMATICS magazine

# MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

EDITORIAL STAFF

EXECUTIVE COMMITTEE

D. H. Hyers, Univ. of So. Calif., Los Angeles 7, Calif.
A. D. Michal, Calif. Inst. of Technology, Pasadena, Calif.
Glenn James, Managing editor, 14068 Van Nuys Blvd., Pacoima, Calif.

W. F. Byrne
Homer V. Craig
René Maurice Fréchet
N. E. Norlund
A. W. Richeson
C. K. Robbins
Nilos Sakellariou
Joseph Seidlin
C. N. Shuster
C. D. Smith
Marion E. Stark
V. Thébault
C. W. Trigg
S. T. Sanders (emeritus)

*Address* editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to some member of the Executive Committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are $3.00, sponsoring subscriptions $10.00, single copies 65¢. Reprints: 25 free to authors; 100 @ $1.25 per page with a minimum of $6.00. All reprints are bound.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, Calif.

Advertisers should contact Homer V. Craig, Univ. of Texas, Austin 12, Texas.

Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

SPONSORING SUBSCRIBERS

Hugh M. Allen
G. A. Baker
E. F. Beckenbach
H. W. Becker
Clifford Bell
Russell M. Caldwell
H. V. Craig
Joseph W. Creely
Paul H. Daus
Alexander Ebin
Theodore M. Edison
Henry E. Fettis
Harry M. Gehman
J. Ginsburg
Merton T. Goodrich
Harold Grant
M. R. Hestenes
Robert B. Herrera
Jerome Hines
Donald H. Hyers
C. J. Jaeger
Glenn James
Robert C. James
John Kronsbein
Lillian R. Lieber
Clyde Love
A. D. Michal
L. R. Robinson
H. A. S. Sharma
C. N. Shuster
H. A. Simmons
M. E. Stark
D. Victor Steed
Paul D. Thomas
E. M. Tingley
H. S. Vandiver

## CONTENTS

*COMING IN JANUARY*

**A NEW TEXT BY WILLIAM L. HART**

# COLLEGE TRIGONOMETRY

## WITH LOGARITHMIC AND TRIGONOMETRIC TABLES

A substantial treatment of plane and spherical trigonometry, of moderate length, incorporating distinctly collegiate viewpoints.

Commences with an introductory chapter on the trigonometric functions of the general angle.

Distinctly emphasizes analytic trigonometry, oriented for application in later mathematics.

Presents a mature and well-rounded treatment of numerical plane trigonometry.

Offers a satisfactory foundation in spherical trigonometry, including a reasonable number of elementary applications.

Includes the author's extensive Logarithmic and Trigonometric Tables, revised by the addition of a six-page table of haversines and their logarithms.

The type is exceptionally legible, the page large and open, and the format generally appealing.

*Text pages:* PLANE TRIGONOMETRY, 151 pp.

COMPLEX NUMBERS AND APPENDIX, 21 pp.

SPHERICAL TRIGONOMETRY, 35 pp.

Tables, 130 pp.

**D. C. HEATH and COMPANY**

College Department

285 Columbus Avenue — Boston 16

# THE LINEAR DIOPHANTINE EQUATION IN TWO UNKNOWNS

J.M. Thomas

1. Introduction. This paper gives a simple method for finding a general solution and the least solutions for the equation

$$ax + by = c \tag{1.1}$$

in which $a$, $b$, $c$ are given positive integers and $x$, $y$ are unknown integers.

A least solution is one for which $x^2 + y^2$ is least.

Limiting the coefficients to positive values is convenient from the standpoint of language, but the methods are directly applicable to the case where one or more of the coefficients are negative and the case $abc = 0$ is trivial.

The treatment does not employ continued fractions, although it is intimately connected (§6) with their theory. The usual procedure in solving (1.1) is to multiply a particular solution of

$$ax + by = 1 \tag{1.2}$$

by $c$. In general, this gives a solution of (1.1) far from the minimum. The particular solution of (1.2) employed is given by the next to the last convergent of the shorter continued fraction determined by $a, b$. That this particular solution is a least solution of (1.2) results from the present treatment and seems to have been unnoticed before, although Lucas [2; 479] has obtained the least solutions of (1.1) from the solution which in the notation of §6 is $(-qc, pc)$ or $(sc, -rc)$.

2. Direct solution. The direct method of trial is suggestive and for the small coefficients usually encountered in the class room seems at least as economical from the standpoint of arithmetic as any other. This method can be based on the simplest geometrical considerations. The totality of real pairs $(x, y)$ satisfying (1.1) is represented by points on a straight line. If point $(x, y)$ is on that line so also is $(x + b, y - a)$. If there is one point with integral coördinates on the line, there is an infinity whose projections on the $x$-axis are equally spaced at distance $b$ apart. Hence, if there is an integral solution, any interval of length $b$ on the $x$-axis contains at least one $x$ which substituted in (1.1) gives an integral $y$. If the end-points of the interval are integers, the above statement remains true for length $b - 1$. Accordingly, let the numbers $0, 1, \cdots, b - 1$ be substituted for $x$ and the corresponding $y$'s be determined. The set of all solutions $(x_0, y_0)$ whose $x$ is on the interval $[0, b - 1]$ is obtained by rejecting the pairs with fractional $y$'s and a general solution is $(x_0 + bt, y_0 - at)$, where $(x_0, y_0)$ runs through the set on $[0, b - 1]$ and $t$ runs through all integers. If $a, b$ are relatively prime, there is

exactly one $(x_0, y_0)$ with $0 \leq x_0 \leq b-1$ and each solution is given exactly once by the general solution described above.

3. A general solution. Rewrite (1.1) in the form

$$\begin{vmatrix} x & -b \\ y & a \end{vmatrix} = c. \tag{3.1}$$

If the integers in the second column are numerically unequal, add the row containing the numerically smaller of them to the other row. Repeat this on the new determinant. Ultimately, the integers in the second column become numerically equal and (3.1) has been transformed into

$$\begin{vmatrix} px + qy & -d \\ rx + sy & d \end{vmatrix} = c, \tag{3.2}$$

an equation equivalent to (3.1) because the steps taken in transforming (3.1) into (3.2) can be reversed by using subtraction instead of addition.

If $d$ does not divide $c$, there is no solution. If $d$ divides $c$, the solution is reduced to that of an equation

$$\begin{vmatrix} px + qy & -1 \\ rx + sy & 1 \end{vmatrix} = c, \tag{3.3}$$

where $c/d$ has been replaced by a new $c$. It is easy to verify that the determinant of the two linear forms in the first column in invariant under the transformation employed. Since in (3.1) that determinant is 1, in (3.3) it is also 1, so that

$$ps - qr = 1. \tag{3.4}$$

If we put

$$px + qy = u, \qquad rx + sy = v, \tag{3.5}$$

integral $(x, y)$ gives integral $(u, v)$ and, because of (3.4), vice versa. Equations (3.3) and (3.5) are equivalent provided $u + v = c$. Hence, a general solution of (1.1) is

$$x = su - qv, \qquad y = -ru + pv, \qquad u + v = c, \tag{3.6}$$

where the last equation means that $u$, $v$ are to be an arbitrary partition of $c$ into two integers.

Any integer which divides $a$ and $b$ divides the integers in the second column of every one of the determinants employed in reaching (3.2) and therefore divides $d$. If (3.2) is transformed back into (3.1), obviously $d$ remains a divisor of the second column. Hence $d$ divides $a$ and $b$. The integer $d$ is therefore the G. C. D. of $a$ and $b$, which is denoted by $(a, b)$.

A necessary and sufficient condition for the existence of a solution is, therefore, that $(a, b)$ divide $c$.

Moreover, any solution $(x, y)$ of

$$\begin{vmatrix} px + qy & -1 \\ rx + sy & 1 \end{vmatrix} = 1$$

because of (3.2) is a solution of

$$(3.7) \qquad ax + by = (a, b).$$

Consequently, transforming the determinant as is done here is an effective way of expressing the G.C.D. of two numbers linearly in terms of those numbers.

4. The brackets and braces. Put

$[\xi]$ = the greatest integer not greater than $\xi$,
$[\xi]'$ = the least integer not less than $\xi$,
$\{\xi\}$ = the least of the integers nearest to $\xi$,
$\{\xi\}'$ = the greatest of the integers nearest to $\xi$.

These quantities are not independent, but satisfy the relations

$$(4.1) \quad [\xi]' = -[-\xi], \qquad \{\xi\} = -\{-\xi\}', \qquad \{\xi\}' = [\xi + \tfrac{1}{2}]$$

and others deducible from them. Employing all four symbols, however, gives symmetry to the results and impetus to the calculations.

The integers $[\xi]$, $[\xi]'$ are equal if and only if $\xi$ is an integer; in the usual (non-integral) case, $[\xi]'$ is the follower of $[\xi]$. The integers $\{\xi\}$, $\{\xi\}'$ are equal except when $\xi$ is half an odd integer; in the exceptional case $\{\xi\}'$ is the follower of $\{\xi\}$.

The symbol $[\xi]$ is of current usage. The symbol $\{\xi\}$ has been used (see, for example, [3; 118]) to denote what is called here $\{\xi\}'$.

5. The least solutions. Let $(\alpha, \beta)$ be the foot of the perpendicular from the origin to the line (1.1). Then $p\alpha + q\beta$, $r\alpha + s\beta$ are the values (in general, not integral) which substituted for $u$, $v$ in the general solution of (1.1) give $(\alpha, \beta)$. The point which is on the line, which has integral coördinates and which is nearest $(\alpha, \beta)$ is also nearest the origin. If $u$ is made equal to an integer nearest $p\alpha + q\beta$ and $v$ is made $c - u$ in the general solution, a least solution results. Least solutions are accordingly given by

$$u = \{p\alpha + q\beta\}, \qquad u = \{p\alpha + q\beta\}'.$$

From the definition

$$(5.1) \quad \begin{aligned} p\alpha + q\beta &= \{p\alpha + q\beta\} + \theta, & -\tfrac{1}{2} < \theta \le \tfrac{1}{2}, \\ r\alpha + s\beta &= \{r\alpha + s\beta\}' - \phi, & -\tfrac{1}{2} < \phi \le \tfrac{1}{2}. \end{aligned}$$

Since $p\alpha + q\beta$, $r\alpha + s\beta$ are the $u$, $v$ of a point on the line, addition of the equations in (5.1) gives

$$\{p\alpha + q\beta\} + \{r\alpha + s\beta\}' + \theta - \phi = c.$$

Hence $\theta - \phi$ is an integer. Multiplying the second inequality in (5.1) by $-1$ and adding it to the first give

$$-1 < \theta - \phi < 1.$$

The only integer satisfying these inequalities is 0 so that $\theta = \phi$ and

$$\{p\alpha + q\beta\} + \{r\alpha + s\beta\}' = c. \tag{5.2}$$

Similarly,

$$\{p\alpha + q\beta\}' + \{r\alpha + s\beta\} = c. \tag{5.3}$$

The least solutions are accordingly given by

$$u = \{p\alpha + q\beta\}, \qquad v = \{r\alpha + s\beta\}'$$

and by

$$u = \{p\alpha + q\beta\}', \qquad v = \{r\alpha + s\beta\}.$$

Theorem 5.1. *The integers $p$, $q$, $r$, $s$ being determined as indicated in §3 and $(\alpha, \beta)$ being the foot of the perpendicular from the origin to (1.1), a solution of (1.1) in integers for which $x^2 + y^2$ is least is*

$$\begin{aligned} x &= s\{p\alpha + q\beta\} - q\{r\alpha + s\beta\}', \\ y &= -r\{p\alpha + q\beta\} + p\{r\alpha + s\beta\}'. \end{aligned} \tag{5.4}$$

*A second least solution, which differs from the first if and only if $p\alpha + q\beta$ (and consequently $r\alpha + s\beta$) is half an odd integer, is obtained by shifting the accent to the other pair of braces.*

An alternative way of stating the condition for two least solutions is: $\alpha$(and consequently $\beta$) is half an odd integer, one diagonal of the determinant $ps - qr$ contains even integers and the other contains odd integers.

6. Continued fractions. It is now convenient to condense the reduction employed in §3 by adding to a row $m$ times the other row, where the multiplier $m$ is the largest integer which leaves the sign of the affected element unaltered; in other words, the first stage is to reduce $b$ mod $a$ or $a$ mod $b$ according as $a < b$ or $b < a$. The coefficients of $(x, y)$ in the various stages are the numerators and denominators of the convergents and the multipliers are the quotients in the longer of the two developments of $a/b$ into a continued fraction: to make the correspondence complete, if $a < b$, a zero-th convergent $1/0$, if $b \leq a$, convergents $0/1$, $1/0$ of orders $-1$ and $0$, respectively, and a final step making one row of the determinant $ax + by$, 0 may be introduced. In particular, $p/q$ is the last even numbered convergent preceding $a/b$ and $r/s$ the last odd numbered convergent preceding $a/b$.

It is to be noted that for relatively prime $a$, $b$ the usual method of finding a particular solution involves getting a single solution of (1.2) from the third convergent counting from the last, whereas our method employs two solutions $(s, -r)$, $(-q, p)$ of (1.2) which can be obtained from the second and third convergents from the last but which are obtained in §3 in a formally different manner and are employed rather differently in setting up the general solution.

7. Least solutions, $c = 1$. Let $(\alpha_1, \beta_1)$ be the foot of the perpendicular from the origin to the line (1.2). From the identity

$$(7.1)\qquad (pa + qb)^2 + (-qa + pb)^2 = (p^2 + q^2)(a^2 + b^2)$$

and $-qa + pb = 1$ it follows that $p\alpha_1 + q\beta_1 \neq 0$, and similarly $r\alpha_1 + s\beta_1 \neq 0$. Since $p, q, r, s, \alpha_1, \beta_1$ are non-negative, from (3.3) with $c = 1$ follow the inequalities

$$(7.2)\qquad 0 < p\alpha_1 + q\beta_1 < 1, \qquad 0 < r\alpha_1 + s\beta_1 < 1.$$

Hence the braces in the right members of system (5.4) are 1, 0 or 0, 1. In the first case, a least solution of $ax + by = 1$ is $(s, -r)$ and in the second, it is $(-q, p)$. The convergent $p/q$, $r/s$ which is farther from $a/b$ gives the solution nearer the origin, except in the trivial case $a = b = 1$, $p = s = 1$, $q = r = 0$, when both $(s, -r)$ and $(-q, p)$ are least solutions.

Theorem 7.1. *The equation*

$$ax + by = 1, \qquad 0 < a, \qquad 0 < b, \qquad a \neq b, \qquad (a, b) = 1$$

*has a unique solution in integers for which $x^2 + y^2$ is least. This solution is*

$$x = (-1)^k f, \qquad y = (-1)^{k+1} e,$$

*if the shorter development of $a/b$ into a continued fraction has $e/f$ and $a/b$ for $(k-1)$-th and $k$-th convergents, respectively.*

8. An example. In solving a particular example it is best to apply the method of solution rather than substitute in formulas (5.4). Consider

$$23x + 16y = 461.$$

$$\begin{vmatrix} x & -16 \\ y & 23 \end{vmatrix} = \begin{vmatrix} x & -16 \\ x+y & 7 \end{vmatrix} = \begin{vmatrix} 3x+2y & -2 \\ x+y & 7 \end{vmatrix} = \begin{vmatrix} 3x+2y & -2 \\ 10x+7y & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} 13x+9y & -1 \\ 10x+7y & 1 \end{vmatrix} = \begin{vmatrix} 13x+9y & -1 \\ 23x+16y & 0 \end{vmatrix}$$

| | | | | | |
|---|---|---|---|---|---|
| Quotients: | 1 | 2 | 3 | 1 | 1 |
| Convergents: | $\frac{1}{1}$ | $\frac{3}{2}$ | $\frac{10}{7}$ | $\frac{13}{9}$ | $\frac{23}{16}$ |

$$\alpha = \frac{23\cdot 461}{785}, \qquad \beta = \frac{16\cdot 461}{785}$$

$$13\alpha + 9\beta = 260 + \frac{123}{785}$$

$$10\alpha + 7\beta = 201 - \frac{123}{785}$$

$$13x + 9y = 260$$

$$10x + 7y = 201$$

The minimum solution (11, 13), determined by the last system, is unique.

The equation $3x + y = 25$ has two least solutions (7, 4), (8, 1).

From the determinant sequence it is easy to read the quotients and convergents for the continued fraction and also the quotients and remainders for the division algorithm.

8. Added in proof. D. H. Lehmer's "A Note on the Linear Diophantine Equation", American Mathematical Monthly, vol. 48 (1941), pp. 240-246 came to my notice too late to be discussed in the appropriate place. His method of finding a particular solution applies elementary transformations to a determinant and so has marked likeness to mine, but there are also marked differences between the two methods.

A conversation with R. J. Levit led me to notice that $(x,y)$ in the general solution are relatively prime if and only if $(u,v)$ are.

## REFERENCES

1. L.E. Dickson, *History of the Theory of Numbers*, vol. 2, Washington, 1920.

2. E. Lucas, *Théorie des Nombres*, Paris, 1891.

3. J.V. Uspensky and M.A. Heaslet, *Elementary Number Theory*, New York, 1939.

Duke University

# COLLEGIATE ARTICLES

Graduate training not required for Reading

# ELEMENTARY CONCEPTS OF FUNCTIONAL MEANS AND DISPERSIONS

John P. Gill

## *INTRODUCTION*

Since the mean or average plays such an important part in all statistics, theoretical and applied, it is well to review some of the elementary concepts concerning this measure. In this country one of the outstanding writers on the theory of the mean was the late Professor Edward L. Dodd under whom this writer had the honor to study several summers. Hence, some of the material that follows deals with the writer's own concepts while some parts (some elementary notions in part I and all of part III) were taken from lectures on the theory of the mean. The last part of this paper deals with a rigorous proof of the mean difference formula* and some applications of this formula to summation problems.

### PART I

The cardinal notion of a mean might be expressed as follows: for each of $N$ numbers, in general different, there is substituted a single number $M$ in a certain relation. In ordinary applications we think of the mean as a measure of central tendency, and generally associate with it a measure of scatter or dispersion either numerical or graphical.

(I) *Definition: A test for any mean function will be to substitute the same constant ($\neq 0$) for each variate and show that the result is that constant; that is, let $N$ be the number of values of the variable $x_i$ $(i = 1, 2, \cdots, N)$, and $C$ is a constant $\neq 0$. Then $F(x_i)$ is a mean function if $F(C) = C$.*

The following common means satisfy the above test: arithmetic, geometric, harmonic, exponential, and the power means. Test (I) will be applied to two of these means as an illustration. The exponential mean may be defined as

$$F(x_i) = \log_b \frac{1}{N} \sum_{i=1}^{N} b^{x_i}, \qquad b > 0 \text{ and } \neq 1.$$

*See Bowley, A.L.: *Elements of Statistics*, Fourth Edition, page 114.

Therefore,

$$F(C) = C.$$

The second illustration will be the test of the power means.

$$F(x_i) = \left\{\frac{1}{N}\sum_{i=1}^{N} x_i^p\right\}^{\frac{1}{p}}, \qquad x_i \geq 0.$$

$$F(C) = C.$$

In the mathematical theory of investments, the equated time $t$ for equitably discharging a series of debts due at different times is a functional mean satisfying test (I). The fundamental equation involving $t$ is

$$\left(\sum_{i=1}^{N} S_i\right)(1+r)^{-t} = \sum_{i=1}^{N} S_i (1+r)^{-x_i}, \qquad (1)$$

where $S_i$ is the sum due $x_i$ periods from to-day, and $r$ is the effective compound interest rate per period. Equation (I) reduces to

$$t = \left\{\log \frac{\sum_{i=1}^{N} S_i}{\sum_{i=1}^{N} S_i (1+r)^{-x_i}}\right\} \div \log(1+r) = F(x_i). \qquad (2)$$

Test (I) gives

$$F(C) = \frac{C \log(1+r)}{\log(1+r)} = C.$$

The approximate formula for $t$ is given as

$$t = \frac{\sum_{i=1}^{N} S_i x_i}{\sum_{i=1}^{N} S_i} = F(x_i), \qquad (3)$$

and test (I) gives $F(C) = C$.

Another formula appearing in the mathematical theory of investments is that for the composite life $t$ of a plant computed under the sinking fund method. The fundamental formula for $t$ is

$$S_{\overline{t}|\text{ at } r} = \left(\sum_{i=1}^{N} W_i\right) \div \sum_{i=1}^{N} \frac{W_i}{S_{\overline{x_i}|\text{ at } r}}, \qquad (4)$$

where $W_i$ is the wearing value of the component part with estimated life $x_i$ periods, $r$ is the effective compound interest rate per period, and

$$S_{\overline{x_i}|\text{ at } r} = \frac{(1+r)^{x_i} - 1}{r} .$$

Equation (4) reduces to

$$t = \left\{\log\left(r \frac{\sum_{i=1}^{N} W_i}{\sum_{i=1}^{N} \frac{W_i}{S_{\overline{x_i}|\text{ at } r}}} + 1\right)\right\} \div \log(1+r) = F(x_i). \qquad (5)$$

According to test (I),

$$F(C) = \{C \log(1+r)\} \div \log(1+r) = C.$$

## PART II

## LEAST SQUARE CONCEPTS

In the preceding discussion of functional means, the median was omitted when reference was made to the common means satisfying test (I) in that this measure of central tendency is associated with discontinuous functions and cannot be handled mathematically. However, if we define the median $A$ as that number which would make

$$\sum_{i=1}^{N} |x_i - A| \qquad (6)$$

a minimum, it can be proved that for an even number of variates $x_i$ there are an infinite number of such solutions for $A$.

As a consequence of this difficulty, for practical purposes various agreements are made as to the value of the median under such conditions; but for further mathematical purposes the theory of least squares supplants (6) above merely supposing that

$$\text{(II)} \qquad \sum_{i=1}^{N} (x_i - A)^2$$

be a minimum. As is well known, (II) is satisfied when

$$A = \frac{\sum_{i=1}^{N} x_i}{N}$$

The theory of least squares affords many functional means according to test (I). For example,

(III)
$$\sum_{i=1}^{N} \left( \frac{x_i - A}{x_i} \right)^2$$

is a minimum when

$$A = \frac{\sum_{i=1}^{N} \frac{1}{x_i}}{\sum_{i=1}^{N} \frac{1}{x_i^2}} = F(x_i).$$

Test (I) gives $F(C) = C$.

It is seen, therefore, that one can introduce the mean by arbitrary definition or derive means by use of least squares theory.

## PART III

### SOME CONCEPTS ASSOCIATED WITH TEST (I)

*Definition: An internal mean M of N variates is one that satisfies the following condition:*

(IV) $$\text{Min}(x_1, x_2, \ldots, x_N) \leq M \leq \text{Max}(x_1, x_2, \ldots, x_N).$$

Definition (IV) implies definition (I), but definition (I) does not imply definition (IV). The reasoning follows.

According to (IV), let

$$M = F(x_i).$$

If
$$x_i = C, \qquad i = 1, 2, \ldots, N.$$

$$\text{Min}(C, C, \ldots, C) \leq M \leq \text{Max}(C, C, \ldots, C),$$

or
$$C \leq M \leq C,$$

and
$$M = C = F(C).$$

On the other hand, definition (I) does not imply definition (IV). This can be shown by a simple example. Let

$$M = \sqrt{\sum_{i=1}^{N} \frac{x_i^2}{N}}.$$

If
$$x_1 = x_2 = \ldots = x_N = C,$$
$$M = \pm C.$$

Should
$$\text{Min}(C, C, \ldots, C) \leq C \leq \text{Max}(C, C, \ldots, C),$$

$-C$ would fail.

*Theorem: Given that $M = F(x_i)$ is a monotonic increasing mean thus satisfying $F(C) = C$ uniquely, then $M$ is an internal mean.*

Proof: Suppose that for $\mu = \text{Max}(x_1, x_2, , , , x_N)$, $M = \mu + C$, $C > 0$, or that $M$ is an external mean. Since $F(x_i)$ is a monotonic increasing function,

$$F(\mu) \geq F(x_i).$$

But for this unique mean

$$F(\mu) = \mu \text{ (only)}.$$

Then

$$\mu \geq F(x_i) = \mu + C, \qquad C > 0, \text{ or}$$

$$\mu \geq \mu + C, \text{ which is impossible.}$$

In a similar manner, suppose $W = \text{Min}(x_1, x_2, , , , , x_N)$, and $M = W - C$, $C > 0$. Now

$$F(W) \leq F(x_i) = M,$$

or

$$W \leq W - C, \qquad C > 0, \qquad \text{which is impossible.}$$

## PART IV

## DISPERSION FUNCTIONS

The function

$$F(x_i) = \frac{\sum_{i=1}^{N} x_i}{N} \pm \sigma_x$$

is a mean function of the $x_i$'s according to definition (I), and can be thought of as a mean range of the variates $x_i$ with respect to the arithmetic average. In a similar manner, other measures as

Median ± interquartile Range

can be thought of as a mean range with respect to the median. Neither of these ranges need be the same just as the arithmetic mean need not coincide with the geometric mean in general.

*Definition: A test for any dispersion function will be to substitute the same constant ($\neq 0$) for each variate and show that the result is zero; that is, $F(x_i)$ is a dispersion function if*

(V) $$F(C) = 0, \qquad C \neq 0.$$

According to definition (V),

$$F(x_i) = \sqrt{\frac{\sum_{i=1}^{N}(x_i - A)^2}{N}}$$

is a dispersion function if $A$ is a mean function of the $x_i$'s according to definition (I).

The function

$$F(x_i) = \text{Max}(x_i) - \text{Min}(x_i)$$

is a dispersion function, while

$$F(x_i) = \frac{\text{Max}(x_i) + \text{Min}(x_i)}{2}$$

is a mean function.

THE MEAN DIFFERENCE

*Definition: Let $D_N$ represent the sum of all possible positive differences of pairs of N variates.*

Assume eight positive variates arrayed. These variates may be individual measures, sampling means or any other statistical measures. Call these variates

$$\begin{gathered} x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \\ x_b > x_a, \text{ for } b > a. \end{gathered} \tag{7}$$

A simple operator will be introduced to facilitate the work that follows.

(VI) *Definition: If $x_a$ and $x_b$ are two positive variates, then*

$$d_b^a = x_a - x_b.$$

From the above definition the following results are obtained:

(VII) $$d_b^a + d_d^c = d_d^a + d_b^c, \text{ and}$$

(VIII) $$d_b^a + d_c^b = d_c^a, \text{ or } d_b^a + d_a^c = d_b^c.$$

Using definition (VI) above, the sum of all possible positive differences between the pairs of the eight variates in (7) can be described as follows:

$$D_8 = \sum_{i=2}^{8} d_1^i + \sum_{i=3}^{8} d_2^i + \sum_{i=4}^{8} d_3^i + \sum_{i=5}^{8} d_4^i + \sum_{i=6}^{8} d_5^i + \sum_{i=7}^{8} d_6^i + d_7^8 .$$

Combining the first terms of each summation; then the second terms in each summation remaining, etc., and applying (VIII) each time, results in

$$D_N = 7d_1^8 + 5d_2^7 + 3d_3^6 + d_4^5 = D_8,$$

by the definition of $D_N$.

(IX) Theorem:

$$D_N = (N-1)d_1^N + (N-3)d_2^{N-1} + \cdots\cdots + hd_{\left|\frac{N}{2}\right|}^{\left|\frac{N}{2}\right|+h},$$

*where* $\begin{cases} h = 1 \text{ when } N \text{ is even, and} \\ h = 2 \text{ when } N \text{ is odd,} \end{cases}$

*and* $\left|\frac{N}{2}\right|$ *is the greatest integer contained in* $\frac{N}{2}$.

Proof: By definition,

$$D_N = \sum_{i=2}^{N} d_1^i + \sum_{i=3}^{N} d_2^i + \sum_{i=4}^{N} d_3^i + \cdots\cdots + \sum_{i=N-1}^{N} d_{N-2}^i + d_{N-1}^N.$$

Form a triangular arrangement of the above summation as follows:

$$D_N = \begin{cases} d_1^2 + d_1^3 + d_1^4 + d_1^5 + d_1^6 + d_1^7 + \cdots\cdots + d_1^N \\ d_2^3 + d_2^4 + d_2^5 + d_2^6 + d_2^7 + d_2^8 + \cdots + d_2^N \\ d_3^4 + d_3^5 + d_3^6 + d_3^7 + d_3^8 + d_3^9 + \cdots + d_3^N \\ d_4^5 + d_4^6 + d_4^7 + d_4^8 + d_4^9 + \cdots + d_4^N \\ d_5^6 + d_5^7 + d_5^8 + d_5^9 + \cdots + d_5^N \\ d_6^7 + d_6^8 + d_6^9 + \cdots + d_6^N \\ d_7^8 + d_7^9 + \cdots + d_7^N \\ \vdots \\ d_{N-3}^{N-2} + d_{N-3}^{N-1} + d_{N-3}^N \\ d_{N-2}^{N-1} + d_{N-2}^N \\ d_{N-1}^N \end{cases} \tag{8}$$

Number the columns from left to right in the triangle of differences (8). Let $C_i$ represent the sum of the differences in the $i$-$th$ column, $i = 1,2,3,,,,,N-1$. Then

$$\text{(X)} \qquad C_i = \sum_{j=1}^{N-i} d_j^{j+i}.$$

By repeated application of (VI), (VII), and (VIII) it can be shown that

$$\begin{aligned} C_1 &= d_1^N \\ C_2 &= d_1^N + d_2^{N-1} \\ C_3 &= d_1^N + d_2^{N-1} + d_3^{N-2} \\ &\;\;\vdots \\ C_{N-3} &= d_1^N + d_2^{N-1} + d_3^{N-2} \\ C_{N-2} &= d_1^N + d_2^{N-1} \\ C_{N-1} &= d_1^N \end{aligned} \qquad (9)$$

The terms in the right side of (9) will add up to $D_N$, provided we can prove

$$C_i \equiv C_{N-i}\,, \qquad i \leq \left|\frac{n}{2}\right|\,, \qquad (10)$$

where $\left|\frac{N}{2}\right|$ is the greatest integer contained in $\frac{N}{2}$.

Consider $i < \frac{N}{2}$, $N$ even. Here (10) becomes

$$\sum_{j=1}^{N-j} d_j^{j+i} \overset{?}{\equiv} \sum_{j=1}^{i} d_j^{j+N-i} \qquad (11)$$

The $d's$ in the right side of (11) have no superscripts in common with the subscripts, for $N + 1 - i > i$ or $\frac{N+1}{2} > i$, since $\frac{N}{2} > i$. Therefore, by (VII), this side can be written

$$\sum_{j=1}^{i} d_j^{j+N-i} \equiv \sum_{j=1}^{i} d_j^{N+1-j} \qquad (12)$$

To prove that the left side of (11) reduces to (12), note that $i + 1 < N - i$, since $i < \frac{N}{2}$ ($N$ is even and $i$ is an integer). It follows, therefore, that the superscripts of the $d's$ in the left side of (11) diminish as follows (reading from right to left after expanding this summation):

$$N, N-1, N-2, , , , , , N-i+1, N-i, , , i+1.$$

As a result, every superscript of the $d's$ fron $N - i$ down through $i + 1$

will have a corresponding subscript among the $d$'s. This follows from the nature of $C_i$. By (VII) certain $d$'s will be eliminated leaving only

$$\sum_{j=1}^{i} d_j^{N+1-j}.$$

For $\frac{N}{2} = i$, (10) becomes $C_{\frac{N}{2}} \equiv C_{N-\frac{N}{2}} = C_{\frac{N}{2}}$.

In a similar manner it can be shown that (11) holds for $N$ odd and where $i < \frac{N-1}{2}$ (or $i < \left|\frac{N}{2}\right|$).

For $N$ odd where $i = \frac{N-1}{2}$

$$C_{\frac{N-1}{2}} \equiv C_{\frac{N+1}{2}}.$$

This is proved directly. By (X)

$$C_{\frac{N-1}{2}} = d_1^{\frac{N+1}{2}} + d_2^{\frac{N+3}{2}} + \cdots\cdots\cdots\cdots\cdots + d_{\frac{N+1}{2}}^{N}, \qquad (13)$$

$$C_{\frac{N+1}{2}} = d_1^{\frac{N+3}{2}} + d_2^{\frac{N+5}{2}} + \cdots\cdots\cdots\cdots\cdots + d_{\frac{N-1}{2}}^{N}. \qquad (14)$$

By (VII),

$$C_{\frac{N-1}{2}} \equiv C_{\frac{N+1}{2}},$$

for the superscript $\frac{N+1}{2}$ in (13) also appears as a subscript. This eliminates one $d$ leaving the same superscripts and subscripts in both (13) and (14).

By successive additions in (9),

$C_1 + C_{N-1}$, $C_2 + C_{N-2}, \ldots\ldots\ldots\ldots, C_{\frac{N-1}{2}} + C_{\frac{N+1}{2}}$ ($N$ odd) or $C_{\frac{N}{2}}$ ($N$ even, there being only a single term $C_{\frac{N}{2}}$).

$$D_N = (N-1)d_1^{N} + (N-3)d_2^{N-1} + \cdots\cdots + hd_{\left|\frac{N}{2}\right|}^{\left|\frac{N}{2}\right|+h},$$

where $h = 2$ for $N$ odd, and $h = 1$ for $N$ even.

(XI) *Definition:* $D = \frac{D_N}{N(N-1)/2}$, *which is a dispersion function according to* (V).

It seems appropriate to associate $D$ with the mode or median in ordinary applications and to use this measure with the functional means for further analytical work.

There follows a computational scheme to determine $D$ in a frequency distribution.

| Mid-point | Frequency | Computational scheme |
| --- | --- | --- |
| 2 | 1 | 1 |
| 4 | 3 | 3 |
| 6 | 5 | 1 4 |
| 8 | 21 | 4 17 |
| 10 | 5 | 1 3 1 |
| | 35 | |

Here $N = 35$, $x_{35} = x_{34} = , = , = x_{31} = 10$,,,,, and $x_1 = 2$. Therefore,

$$D = \frac{(34)(8) + (32 + 30 + 28)(6) + (26)(4) + (24 + 22 \quad 20 + 18)(2) + 0}{35 \times 17},$$

and

$$D = \frac{1,084}{595} = 1.8 .$$

*********************************

*Corollary 1: The sum of all the possible positive differences of the first N integers is*

$$D_N = (N-1)^2 + (N-3)^2 + \cdots\cdots + h^2, \qquad \begin{array}{l} h = 1,\ N \text{ even}, \\ h = 2,\ N \text{ odd}. \end{array}$$

Here $x_N$ is $N$ and $x_1 = 1$. Therefore, $d_1^N = (N-1)$, $d_2^{N-1} = N-3, \cdots$

$\cdots, d_{\left|\frac{N}{2}\right|}^{\left|\frac{N}{2}\right| + h} = h$. Applying (IX), we have the proof.

*Corollary 2: The sum of all possible positive differences of the first N even integers is*

$$D_N = 2\,\{(N-1)^2 + (N-3)^2 + \cdots\cdots + h^2\}, \qquad \begin{array}{l} h = 1,\ N \text{ even}, \\ h = 2,\ N \text{ odd}. \end{array}$$

Here, $x_N = 2N$ and $x_1 = 2$. $d_1^N = 2N - 2$, $d_1^{N-1} = 2N - 2 - 4 = 2(N-3)$, etc. Applying (IX), we have the proof.

*Corollary 3: The sum of all possible positive differences of the first N integers each integer squared is*

$$D_N = (N+1)\,\{\,(N-1)^2 + (N-3)^2 + \cdots + h^2\,\}, \qquad \begin{array}{l} h = 1,\ N \text{ even}, \\ h = 2,\ N \text{ odd}. \end{array}$$

Proof:

$$x_N = N^2 \text{ and } x_1 = 1^2,\ x_2 = 2^2, \text{ etc.}$$

$$d_1^N = N^2 - 1 = (N-1)(N+1),$$

$$d_2^{N-1} = (N-1)^2 - 2^2 = (N-1-2)(N-1+2), \cdots\cdots\cdot, \text{ and}$$

$$d_{\left|\frac{N}{2}\right|}^{\left|\frac{N}{2}\right|+h} = \left\{ \left(\left|\tfrac{N}{2}\right| + h\right)^2 - \left|\tfrac{N}{2}\right|^2 \right\} = \left(\left|\tfrac{N}{2}\right| + h - \left|\tfrac{N}{2}\right|\right)\left(\left|\tfrac{N}{2}\right| + h + \left|\tfrac{N}{2}\right|\right)$$

$= h(h + N)$, if $N$ is even, *i.e.*, $h(1 + N)$.

If $N$ is odd, the term

$\left|\frac{N}{2}\right| + h + \left|\frac{N}{2}\right|$ is equivalent to $\frac{N-1}{2} + h + \frac{N-1}{2} = h + N - 1.$

But if $N$ is odd, $h$ is 2 and this reduces to $2 + N - 1 = N + 1$. Applying (IX), we have the proof.

## PAPERS ON MEANS BY EDWARD L. DODD

"Means or Averages and Their Uses," *University of Texas Bulletin*, February 8, 1936, No. 3066, pp. 19–34. A presentation for non-mathematicians.

"Index Numbers and Regression Coefficients as Means, Internal and External," abstract in Cowles Commission for Research in Economics, *Report of Third Annual Research Conference on Economics and Statistics*, . . . 1937, pp. 13–14.

"Necessary and Sufficient Conditions for Means Arising from the location and scaling of Frequency Functions," *ibid.*, pp. 15–16.

"Internal and External Means Arising from the Scaling of Frequency Functions," *Annals of Mathematical Statistics*, vol. 8, March, 1937, pp. 12–20.

"The Median, Quartiles, and other Positional Means," *American Mathematical Monthly*, Vol. 45, May, 1938, pp. 302–306.

"Interior and Exterior Means Obtained by the Method of Moments," *Annals of Mathematical Statistics*, Vol. 9, September, 1938, pp. 153–157.

"The Substitute Mean and Certain Subclasses of the General Mean," *Annals of Mathematical Statistics*, Vol. 11, June 1940, pp. 163–175. Includes a bibliography of 37 items.

"The Complete Independence of Certain Properties of Means," *Annals of Mathematics*, Vol. 35, 1934, pp. 740–747.

"Regression Coefficients as Means of Certain Ratios," *American Mathematical Monthly*, Vol. 34, 1937, pp. 306–308.

"Some Generalizations of the Logarithmic Mean and of Similar Means of Two Variates which Become Indeterminate when the Two Variates are Equal," *Annals of Mathematical Statistics*, Vol. 12, No. 4, December, 1941, pp. 422–428.

Florida State University

# TEACHING OF MATHEMATICS

*Edited by*

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topics related to teaching, are invited. Papers on any subject in which you, as *a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

## SOME HISTORIC AND PHILOSOPHIC ASPECTS OF GEOMETRY

Roger Osborn

It is important for a teacher of any subject to have as much knowledge as possible about the history and philosophy of his subject. It seems that many teachers of mathematics in the modern day have neglected this phase of their work. This is true particularly of teachers of the more elementary courses in geometry. It is the purpose of this paper to stimulate interest in the history and philosophy of geometry by considering the birth and growth of non-Euclidean geometry. Teachers of mathematics who may find this paper of interest will also find that further study can enrich their classroom discussions and that their students may develop a greater appreciation for the subject of geometry.

The high point in the philosophy of geometry came with the realization that there are two kinds of geometries - mathematical and physical. In the history of geometry, the realization of this difference marked the one great advance made in geometry in nearly twenty-five centuries. This will be noted in the summaries of both the history and philosophy of non-Euclidean geometry that follow.

The birthplace of geometry, according to Herodotus, was ancient Egypt. Its very beginning can be traced as far back as B.C. 3000. There was developed the science of earth-measuring. The architects, too, used geometry. The Greeks developed these measuring procedures into a deductive science mainly through the work of Euclid. Euclid made many contributions to the subject, but his greatest was that of collecting and systematizing a great amount of work which had been done already. Geminos of Rhodes (B.C. 70) began to think seriously of the trouble involved should one try to demonstrate Euclid's parallel postulate (postulate 5 given later). Euclid himself seems to have been aware of these difficulties. Ptolemy tried to prove the Euclidean postulate, but met with no success.

An Arab, Nasir-Eddin, tried during the thirteenth century to improve the problem of parallelism. His work was printed in 1594 in Rome and was presented by John Wallis in the late seventeenth century to mathematicians at Oxford. They regarded the work as having accomplished nothing, but it did reopen speculation with regard to the subject of parallelism. G. Saccheri treated the problem in the early eighteenth century. He seems to have been the first to propose a non-Euclidean parallel postulate (rather than trying to prove the old one), but he did not pursue the subject thoroughly. He arrived at no contradiction from his assumption, but he did not take this cue which might have led to the development of a non-Euclidean geometry a century early. Many other great mathematicians worked on this problem - John Lambert, J.L. Lagrange, A.M. Legendre, C.F. Gauss, F.K. Schweikart, etc. Some of these men even had embryonic systems of non-Euclidean geometry.

To two men, working independently of each other, and without the knowledge of the discoveries of Gauss and others, goes the honor of first creating a complete non-Euclidean geometry. These two men were Janos Bolyai, a young officer in the Hungarian army, and N.I. Lobachevski, Professor of Mathematics at the University of Kazan, Russia. The work of these two men was more than investigative, it was constructive.

Lobachevski, feeling the fruitlessness of the work of men during the 2000 years preceding his time, was led to the discovery of a non-Euclidean geometry through his own fruitless researches into the theory of parallels. He first published his work in 1829 in the Russian language. He later translated it into French and German in an attempt to popularize it. J. Bolyai was led to the discovery in a very similar manner and at about the same time. He transmitted his work to his father for publication as an appendix to a book on geometry by his father. It was finally published in 1832. Bolyai's work appeared first in Magyar. With the two original works in such obscure languages, they attracted practically no attention until they were read and written about by Beltrami in 1868. It is a matter of record that Gauss knew of the work of the younger Bolyai and praised it in his correspondence with the elder Bolyai, but, regrettably, he did not use his great influence to gain recognition for the new work.

The works of Bolyai and Lobachevski fall into a formative period. The next period has been characterized as the determinative period. To this period belong the works of E. Beltrami and B. Riemann. In addition to his work of popularizing the original works on the subject, Beltrami did work of his own by conceiving surfaces on which Euclidean geometry does not prevail, and by so doing established the new subject as a real geometry by making it intuitively true as well as logically consistent. Riemann not only felt that Euclid's parallelism axiom was worthy of consideration but also his axiom regarding the infinite extent of lines (axiom 2). Thus he was able to create two types of non-Euclidean geometry.

Much work has been done since the time of Riemann, but it has been more in the nature of extending the results of previous work. Much work, too, has been done in the field of the philosophy of geometry.

These later efforts have been classified as belonging to the elaborative period.

This has been a brief history of the subject of Euclidean (and later, non-Euclidean) geometry. In order to understand better the meanings and significances of non-Euclidean geometry, one should know some of the likenesses and differences of the two.

Euclid formed his geometry on five basic assumptions concerning magnitude and five basic postulates. These five postulates are, essentially,

(1) It is possible to draw a straight line joining any two points,

(2) A straight line segment may be extended without limit in either direction,

(3) It is possible to draw a given circle with a given center through a given point,

(4) All right angles are equal, and

(5) If two straight lines in a plane meet another straight line in the plane so that the sum of the interior angles on the same side of the latter straight line is less than two right angles, then the two straight lines will meet on that side of the latter straight line.

A non-Euclidean geometry is generally understood to be one built up without the aid of the Euclidean parallel postulate (number 5) while at the same time it contains as assumption about parallels incompatible with that of Euclid.

In the construction of his non-Euclidean geometry, Lobachevski used the parallel postulate: with respect to a given straight line, all others in the same plane may be divided into two classes, those which cut the given line, and those which do not cut it; a line which is the limit between these two classes is called parallel to the given straight line. This assumption with regard to parallelism led Lobachevski to the conclusion that there are two parallels to a line through a given point not on the line. He deduced from this that the angles of a triangle must be less than or equal to two right triangles. The former case gives a non-Euclidean geometry and the latter a Euclidean geometry.

J. Bolyai constructed a non-Euclidean geometry in which the methods and results were the same. The main technical difference between the works of the two men lies in the fact that Bolyai assumed that "more than one" parallel can be drawn to a given line through a given point outside the line.

Riemann formed a non-Euclidean geometry different from those of Bolyai and Lobachevski. His was formed from a consideration of two postulates differing from those of Euclid. He rejected postulates (2) and (5). In rejecting (2), he formed one which postulated that a line may be finite in length. In his system, postulate (5) is not needed since there are no parallels to a given straight line through a given point.

In the latter part of the nineteenth century it was shown that the difference between the various geometries is metrical and arises from different definitions of magnitude, of distance, etc. A convenient nomenclature introduced by F. Klein listed the three geometries as as hyperbolic (Lobachevski and Bolyai), elliptic (Riemann), and parabolic (Euclid).

A readable exposition of the foundations of non-Euclidean geometry and examples of the types of space considered may be found in *The Encyclopedia Americana.* A list of the definitions of terms may be found in *The Mystery of Space* by Robert T. Browne (New York, E.P. Dutton and Co., 1919). Following is a brief list of some of the results obtained in non-Euclidean geometries:

(1) The sum of the angles of a triangle is either greater than or less than two right angles.

(2) The angle sums of two triangles of equal area are equal.

(3) Similar triangles of different sizes are impossible.

(4) If two equal perpendiculars are erected to the same straight line, their distance apart increases with their length.

(5) A line, every point of which is equally distant from a given straight line, is a curved line.

(6) Any two lines which do not meet, even at infinity, have one common perpendicular which measures their minimum distance.

(7) Lines which meet only at infinity are parallel.

The most important difference between all the geometries is that of the difference of philosophy. This is because great changes in thought arise from new philosophies. The first geometry dealt with physical relations. The Greeks developed the methods and devices of measuring into a deductive science. In any deductive system the postulates are of utmost importance and hence must be selected with care. Fuclid must have been aware of this since he selected postulates which he considered to be *intuitively* true in general. His parallel postulate, though, differed from the others in this respect, and it is certain that he realized this.

In the study of Euclidean geometry, three problems presented themselves regarding parallelism: (1) was the parallel postulate deducible from the other assumptions of Euclidean geometry, (2) if not, was it an assumption demanded by the facts of experience in order that further propositions describe the space in which we live, and (3) were both it and assumptions incompatible with it consistent with other assumptions of Euclidean geometry so that the Euclidean parallel postulate was a special case used because it was more convenient? Euclid himself answered the first question negatively and assumed the answer to the second was also negative. The third question was answered with a definite affirmative by Bolyai and Lobachevski. By logical processes, a negative answer could then be obtained to the first questions. These same three questions seem to divide the discussion of the philosophy of geometry. That is, there must be considerations of the nature of geometry with regard to space concepts and also the nature with regard to truth, falsity, or convenience.

The answers to the problems presented by these considerations lie entirely within the concept of geometry. Actually, there are two types of geometry - the mathematical and the physical. If this distinction is made and explained, then any further philosophical question can be answered only after the question has been placed in the proper category. The difference between the two types of geometry - mathematical and physical - did not become apparent until a non-Euclidean geometry was discovered. It has been shown that logically non-Euclidean geometry is free from contradictions if Euclidean geometry is free, and mathematicians investigate properties of either. Physicists, though, ask which is true, which represents nature. In the mathematical geometry, there is no truth or falsity. Modern mathematics presents us with a series of possible forms of spatial relations, but it does not tell us which one will enable us best to interpret the physical world. Physicists are interested in the interpretations of geometry, and here there may be truth or falsity in the applications to physical phenomena. Hence it is the position of the physicist rather than the mathematician to determine truth or falsity of application to the best degree of confirmation. Classical physics said the nature of space was Euclidean; modern physics says it is non-Euclidean, but that Euclidean concepts are good enough for many purposes. Euclidean geometry is not so certain in its application to space as it is utilitarian; but non-Euclidean geometry is even less certain of application and hence more lacking in utilitarianism.

The process of extablishing mathematical geometry as a subject completely independent of physical relations was laborious.

The development of non-Euclidean geometry and the general acceptance of its validity was considerably dependent upon conceptions of space. The slowness with which the early writings were accepted is attributable to a great extent to the fact that the new geometry was contrary to the Kantian philosophy of space. Kant proposed the doctrine that space is a subjective intuition, a necessary presupposition to every experience. This was the accepted doctrine at the time the non-Euclidean geometries were developed. Another commonly accepted idea which slowed the acceptance of non-Euclidean geometries was that they implied (if the physical would could be described by them) that a space constant (a linear magnitude) existed which was determined in itself. This was inconceivable to thinkers of that time, though it is not now held to be inconceivable.

Mathematical geometry is now considered to be a subject which asserts only that certain consequences follow from certain hypotheses whether or not the entities described in the hypotheses actually exist. Physical geometry, that is, the applications of mathematical geometry to the physics of space still must be classified as convenient or inconvenient. The statements of mathematical geometry are purely formal, but the interpretations placed on these statements are factual and descriptive.

The discovery of non-Euclidean geometry has thus led to new space concepts as well as having settled a mathematical dispute centuries old. Physical applications have been found to utilize the discovery. The final issue of non-Euclidean geometry, though, is not in its utility nor its inclination to a new outlook but it is solely in the possibilities in the domain of analytic thought which it has opened to view.

University of Texas

# INTEGRAL EQUATIONS AND FUNCTIONALS

Aristotle D. Michal

*Introduction.* It would be difficult to think of any two topics in mathematical analysis more central and more widely studied during the last fifty years than the theory of integral equations and functionals. Here we are using the word functional as a *noun* and not as an adjective. A functional is a generalization of the notion of a function of a finite number of numerical variables while an integral equation is, as its name suggests, an equation in which the unknown, say a function of a numerical variable, occurs under an integral.

The early history of integral equations goes back to the special integral equations studied by several mathematicians of the late eighteenth and early nineteenth century - Laplace, Fourier, Poisson, Abel and Liouville - while the pioneering systematic investigations go back to the late nineteenth and early twentieth century work of Volterra, Fredholm and Hilbert.

The precise definition of a functional will be given later. To give an approximate idea of the flavor of the subject, it is instructive to point out a few simple examples of a functional with which the reader is familiar in his elementary calculus courses.

The length of a curve depends for its value on the curve and so it is a functional of the curve. The area enclosed by a closed plane curve depends on the closed curve for its value and hence it is a functional of a closed plane curve. The volume enclosed by a closed surface in space is a functional of the closed surface. Finally, the integral of a function depends for its value on the function and so it is a functional of the function. We can continue with many other more complicated examples. However, the above illustrations will suffice to show clearly that some of the special functionals have a history going back to the founders of the differential and integral calculus and even as far back as the ancient Greek mathematicians. The notion of the variation of an integral as first employed by Euler and Lagrange in their treatment of the calculus of variations can be viewed as a first, but partially successful, attempt to define a differential of a functional for the special functionals of the calculus of variations. The notion of a differential of a functional is very fundamental in the modern theory of functionals.

In 1887, Volterra published a series of famous papers in which he singled out the notion of a functional and pioneered in the development of a theory of functionals. During the next decade, Volterra laid the foundations for his theory of linear integral equations while at the turn of the century, Fredholm gave the fundamentals of the Fredholm integral equation theory, (first used by him in the solution of the Dirichlet problem) a theory which generalizes to the continuous domain

Cramer's solution of linear algebraic equations as ratios of determinants. Fredholm presented the fundamentals of the Fredholm integral equation theory in a paper published in 1903 in the Acta Mathematica. This paper became famous almost overnight and soon took its rightful place among the gems of modern mathematics. Hilbert followed Fredholm's famous paper with a series of papers in the Nachrichten of the Göttingen Academy. In them he developed his theory of linear integral equations and the accompanying development of what is now known as the (classical) Hilbert space theory.

Let us now consider what is technically known as a Volterra linear integral equation of the second kind

$$f(x) = y(x) + \int_a^x K(x,s)y(s)ds. \tag{1}$$

The function $f(x)$ is given and is assumed, say, continuous in the closed finite interval $a \leq x \leq b$. The function $K(x,s)$, called the *kernel* of the integral equation, is given and is assumed, say continuous in the square $a \leq x,\ s \leq b$. The unknown in the equation is the function $y(x)$.

We shall illustrate the general method of solving such an integral equation by the following instructive special equation

$$y_0 = y(x) + \int_a^x y(s)ds. \tag{2}$$

In this integral equation, the known function $f(x)$ is a constant $y_0$, and the known kernel $K(x,s) \equiv 1$ in $(a,b)$. Let us, for the moment, assume that there exists a continuous solution $y(x)$ of (2), i.e., assume there exists a function $y(x)$ continuous in $a \leq x \leq b$ which when substituted in the integral equation (2) will yield an identity in $x$ for all $x$ in $a \leq x \leq b$. Let us write (2) in the equivalent form

$$y(x) = y_0 - \int_a^x y(s)ds. \tag{3}$$

Since $y(x)$ satisfies the equation (3), we can *substitute* for $y(s)$ under the integral sign an expression to which it is equal, i.e.,

$$y(s) = y_0 - \int_a^s y(s_1)ds_1.$$

We thus obtain after evident simplifications

$$y(x) = y_0 - y_0\int_a^x ds + \int_a^x ds \int_a^s y(s_1)ds_1$$

or

$$y(x) = y_0 - y_0(x-a) + \int_a^x ds \int_a^s y(s_1)ds_1. \tag{4}$$

Again if we substitute for $y(s_1)$ its equal as given by (3) we obtain after evident simplifications

$$y(x) = y_0 - y_0(x-a) + y_0\frac{(x-a)^2}{2!} - \int_a^x ds \int_a^s ds_1 \int_a^{s_1} y(s_2)ds_2. \tag{5}$$

Continuing indefinitely in this manner of successive substitutions, we are *led* to the following expansion of the assumed solution $y(x)$ of (2)

$$y(x) = y_0 + y_0\left[-(x-a) + \frac{(x-a)^2}{2!} - \frac{(x-a)^3}{3!} \ldots\right],$$

so that

$$y(x) = y_0 e^{-(x-a)}. \tag{6}$$

But by inserting (6) in (2), we see readily that (6) satisfies the integral equation (2). To prove that (6) is the unique continuous solution $y(x)$ in $(a,b)$ of the integral equation (2), we assume that there exists another continuous solution $z(x)$ of (2) in $(a,b)$ such that $z(x) \not\equiv y(x)$ in $(a,b)$. By hypothesis $y(x)$ satisfies (2) and we also have

$$y_0 = z(x) + \int_a^x z(s)ds. \tag{7}$$

Subtracting corresponding sides of (2) and (7) we find

$$0 = [z(x) - y(x)] + \int_a^x [z(s) - y(s)]ds. \tag{8}$$

Define $w(x) = z(x) - y(x)$ and thus we can write (8) in the form

$$w(x) = -\int_a^x w(s)ds. \tag{9}$$

Since $w(x)$ is continuous in $(a,b)$, let $M$ be the maximum of $|w(x)|$ as $x$ varies over the interval $(a,b)$. We thus obtain from (9), the inequality $|w(x)| \leq M(x-a)$ for $a \leq x \leq b$. If we use (9) again we obtain

$$|w(x)| \leq \int_a^x M(s-a)ds \leq M\frac{(x-a)^2}{2!}.$$

Continuing in this way we obtain by induction

$$|w(x)| \leq M \frac{(x - a)^n}{n!} \text{ for all } n = 0, 1, 2, \ldots .$$

Hence $M = 0$. This means that $w(x) \equiv 0$ in $(a,b)$ since $w(x)$ is continuous. Hence $z(x) \equiv y(x)$ in $(a,b)$, which is contrary to our supposition that $z(x) \not\equiv y(x)$ in $(a,b)$. This implies that (6) *is the unique continuous solution in (a,b) of the Volterra integral equation* (2).

Under the restrictions on the known functions $f(x)$ and $K(x,s)$, *the Volterra integral equation of the second kind* (1) *has a unique continuous solution in* $(a,b)$ *and is given by the formula*

$$(10) \qquad y(x) = f(x) + \int_a^x k(x,s)f(s)ds,$$

*where* $k(x,s)$, *called the resolvent kernel of* $K(x,s)$, *is continuous in* $a \leq x,\ s \leq b$ *and has the infinite expansion*

$$(11) \qquad k(x,s) = -K_1(x,s) + K_2(x,s) - K_3(x,s) + \ldots .$$

*In this expansion the functions* $K_i(x,s)$ *are defined inductively in the following manner*

$$(12) \qquad K_1(x,s) = K(x,s),\ K_i(x,s) = \int_s^x K(x,t)K_{i-1}(t,s)dt$$

*for* $i > 1$.

The method of proof can be patterned after the same general method of successive substitutions that was used in treating the special Volterra integral equation (2).

There are two particularly interesting examples of functionals that are defined by (10) and (11) in connection with the above theorem on Volterra integral equations of the second kind. To elucidate this point we now have to give a formal definition of a functional. To do this it is convenient to make the definition by first giving a definition of the general notion of a function $F(x)$ whose independent variable $x$ ranges over any fixed class $C_1$ of elements (not necessarily numerical) and whose values are contained in any fixed class $C_2$ of elements. The class $C_2$ may not necessarily be distinct from $C_1$. $F(x)$ *is a function on* $C_1$ *to* $C_2$ *if to each element* $x$ *in the class* $C_1$ *there corresponds one and only one element called value of the function, of the class* $C_2$. If to some $x$ in $C_1$ there corresponds more than one element of $C_2$, we can speak of a multiple-valued function. It is to be noted that the values of the function need not exhaust the class $C_2$.

**Definition of a functional.** *A functional is a function on* $C_1$ *to* $C_2$ *whenever* $C_1$ *is itself a class of functions of a finite number of numerical variables while* $C_2$ *is another class of functions of a finite* *number of numerical variables.*

In a real function $F(x)$ of a real variable $x$, the independent variable $x$ is a numerical variable ranging over a class of numbers while the dependent variable is also a numerical variable ranging over a class of numbers, i.e., the values of $F(x)$. In a functional $F(x)$, the independent variable $x$ ranges over a class $C_1$ of functions of a finite number of numerical variables, say $t_1, \ldots, t_r$, so that a fixed value of the independent variable $x$ will be a fixed *function* in $C_1$ of the $r$ numerical variables $t_1, t_2, \ldots, t_r$. The dependent variable in the functional $F(x)$ also ranges over a set $S$ of functions which is contained in the class $C_2$ of *functions* of a finite number of numerical variables, say $u_1, u_2, \ldots, u_s$. Thus a fixed value of the functional $F(x)$ will be a fixed *function* in $S$ of the $s$ numerical variables $u_1, u_2, \ldots, u_s$. The particular "degenerate" case in which the values of the functional $F(x)$ are real numbers is also envisaged. This corresponds to the case in which $S$ consists of constant real valued functions.

We are now in a position to discuss the expansions (10) and (11) that occurred in the theory of the Volterra integral equation of the second kind. Let $C_1$ be the class of all real continuous functions $K(x,s)$ of two real variables $x$ and $s$ in the square $a \leq x, s \leq b$. Now for each $K(x,s)$ in $C_1$, the resolvent kernel $k(x,s)$ defined by the expansion (11), is also a continuous function of $x$ and $s$ for $a \leq x, s \leq b$. Hence to each $K(x,s)$ in $C_1$ there corresponds a $k(x,s)$ also in $C_1$. Hence the resolvent kernel $k(x,s)$ is a functional of the kernel $K(x,s)$. In this particular functional, the independent variable $K(x,s)$ ranges over $C_1$ while the dependent variable of the functional has also values in $C_1$. From the formula (10) for the solution $y(x)$ of a Volterra integral equation (1) and from what we have just remarked about the resolvent kernel $k(x,s)$ it follows that the solution $y(x)$ is a functional of the known function $f(x)$ and the kernel $K(x,s)$.

The Fredholm theory of integral equations is one of the most beautiful chapters in modern mathematics. We shall center our attention on the *linear Fredholm integral equation of the second kind*

$$f(x) = y(x) + \int_a^b K(x,s)y(s)ds. \tag{13}$$

The only difference from the form of the Volterra integral equation (1) is that now the upper limit of the integral is the constant $b$, the end point of the closed interval $(a,b)$. We shall make the same restrictions, for the sake of simplicity, on the given function $f(x)$ and $K(x,s)$ as those made in connection with the Volterra equation (1). It is possible to develop a theory of the integral equation (13), similar to the one sketched for the equation (1) with the variable upper limit if the product of the maximum of $|K(x,s)|$ multiplied by the length of the interval $(a,b)$ is less than 1. This, however, is a very stringent restriction that is not made in the Fredholm theory.

To state the fundamental result for the Fredholm equation (13), we need to define two functionals: The *Fredholm ''determinant''* and the *first Fredholm ''minor''*. By the Fredholm determinant $D[K(x,s)]$ of the kernel $K(x,s)$, we mean the functional defined by the following infinite expansion

$$(14)\qquad D[K] = 1 + \int_a^b K(s,s)ds + \frac{1}{2!}\int_a^b\int_a^b \begin{vmatrix} K(s_1,s_1), & K(s_1,s_2) \\ K(s_2,s_1), & K(s_2,s_2) \end{vmatrix} ds_1 ds_2 + \dots + \frac{1}{n!}\int_a^b \cdots \int_a^b \begin{vmatrix} K(s_1,s_1),\dots, & K(s_1,s_n) \\ K(s_2,s_1),\dots, & K(s_2,s_n) \\ \dots\dots & \dots\dots \\ K(s_n,s_1),\dots, & K(s_n,s_n) \end{vmatrix} ds_1\dots ds_n + \dots + \dots$$

while the first Fredholm minor $D[K|x,s]$ is defined by

$$(15)\qquad D[K|x,s] = K(x,s) + \int_a^b \begin{vmatrix} K(x,s), & K(x,s_1) \\ K(s_1,s), & K(s_1,s_1) \end{vmatrix} ds_1 + \frac{1}{2!}\int_a^b\int_a^b \begin{vmatrix} K(x,s), & K(x,s_1), & K(x,s_2) \\ K(s_1,s), & K(s_1,s_1), & K(s_1,s_2) \\ K(s_2,s), & K(s_2,s_1), & K(s_2,s_2) \end{vmatrix} ds_1 ds_2 + \dots + \dots .$$

The expansion (14) converges absolutely for every continuous $K(x,s)$. The expansion (15) is uniformly and absolutely convergent for every continuous $K(x,s)$ in $a \leq x,\ s \leq b$.

**Fredholm's First Fundamental Theorem**

*If the Fredholm determinant* $D[K] \neq 0$, *then the integral equation* (13) *has one and only one continuous solution* $y(x)$ *in* $(a,b)$ *and is given by*

$$(16)\qquad y(x) = f(x) + \int_a^b k(x,s)f(s)ds,$$

*where the resolvent kernel* $k(x,s)$ *of* $K(x,s)$ *is defined by*

$$(17)\qquad k(x,s) = -\frac{D[K|x,s]}{D[K]}.$$

The Fredholm integral equation (13) is a *non*-homogeneous linear integral equation. The *homogeneous* Fredholm linear integral equation

$$(18)\qquad y(x) + \lambda\int_a^b K(x,s)y(s)ds = 0$$

has an extensive and elaborate theory. Many boundary value problems in linear differential equations can be formulated as problems in such homogeneous Fredholm integral equations. The need for a numerical parameter $\lambda$ - possibly a complex number - will presently become clear. Since $f(x) \equiv 0$ in $(a,b)$ in equation (18), it is clear from the formula (16) that the unique continuous solution $y(x)$ of (18) is the trivial $y(x) \equiv 0$ in $(a,b)$ if the Fredholm determinant of the kernel $\lambda K(x,s)$ is not zero. In other words *if the numerical parameter* $\lambda$ *is such that* $D[\lambda K(x,s)] \neq 0$, *then the homogeneous Fredholm integral equation* (18) *has no other continuous solution than the trivial one* $y(x) \equiv 0$ *in* $(a,b)$. Hence, if we are to have a non-trivial solution of the equation (18), it is necessary that the numerical parameter $\lambda$ be a root of the transcendental equation $D[\lambda K(x,s)] = 0$. Values of the parameter $\lambda$ which satisfy this equation are called *characteristic values of the kernel* $K(x,s)$. It can be shown that the resolvent kernel of $\lambda K(x,s)$ is a meromorphic function of the complex variable $\lambda$ and that a characteristic value of a kernel $K(x,s)$ is necessarily a pole of the resolvent kernel of $\lambda K(x,s)$. The following theorem was proved by Fredholm in his fundamental 1903 paper.

**Second Fredholm Theorem.** *If* $\lambda_0$ *is a characteristic value of the kernel* $K(x,s)$, *then the homogeneous Fredholm equation*

$$y(x) + \lambda_0 \int_a^b K(x,s)y(s)ds = 0$$

*has a finite number of linearly independent solutions.*

The Fredholm determinant in the Fredholm theory of integral equations plays a role analogous to the role of the determinant of the coefficients of a system of $n$ linear algebraic equations in $n$ unknowns. In fact, Hilbert demonstrated in his 1904 paper that the Fredholm determinant is the limit of the determinant of a certain system of linear algebraic equations. The first Fredholm theorem is analogous to Cramer's solution of a system of linear algebraic equations when the determinant of the coefficients is not zero. Fredholm's second theorem is analogous to the theorem that a system of $n$ linear homogeneous algebraic equations in $n$ unknowns has a finite number of linearly independent solutions if the determinant of the coefficients of the equations is zero.

The question of the existence of characteristic values in a homogeneous Fredholm integral equation is important because, clearly, the equation can not have any non-trivial solutions unless $\lambda$ is a characteristic value of the kernel $K(x,s)$. Hilbert's point of view in his integral equation theory was that of quadratic forms in an infinite number of variables in which the symmetry of the coefficients of the quadratic form played an essential role. Later in the Hilbert-Schmidt theory developed by Schmidt in his 1905 thesis, it was proved that a continuous *symmetric* kernel $K(x,s) \not\equiv 0$ has at least one real characteristic value. This singles out the class of homogeneous Fredholm integral equations with symmetric kernels as worthy of special study. In fact it is a matter of record that the whole theory of Fredholm integral equations with symmetric kernels attained a relatively high degree of perfection as early as the first decade of our century. Self-adjoint boundary value problems for ordinary and partial differential equations can be formulated as Fredholm integral equations with symmetric kernels.

Although the theory of *linear* integral equations can be developed far without any explicit use of the theory of functionals, the theory of non-linear integral equations requires several chapters of the theory of functionals for its deep study. A few remarks on these matters will be included later, after a few fundamental notions of the theory of functionals have been discussed.

To give a definition of a continuous functional it is necessary to delimit clearly the class of functions for the independent and dependent variables of the functional. For example, take $C_1$ to be the class of all real continuous functions $x(s)$ of a real variable $s$ over the fixed closed finite interval $(a,b)$. Let us use the notation $||x(s)|| \overset{\text{def}}{=} \max_{a \leq s \leq b} |x(s)|$, and call a neighborhood of $x_0(s)$ of $C_1$, all $x(s)$ of $C_1$ such that $||x(s) - x_0(s)|| < \eta$ for some $\eta > 0$. A functional $F(x)$ on a neighborhood of $x_0(s)$ of $C_1$ to $C_1$ is said to be *continuous at* $x(s) = x_0(s)$ of $C_1$, if given any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $||F(x_0 + y) - F(x_0)|| < \epsilon$ for all $y(s)$ of $C_1$ that satisfy the inequality $||y(s)|| < \delta(\epsilon)$.

A *linear functional* on $C_1$ to $C_1$ is a functional $l(x)$ on $C_1$ to $C_1$ with the two properties: $l(x_1 + x_2) = l(x_1) + l(x_2)$ for all $x_1(s)$, $x_2(s)$ of $C_1$; (b) $l(x)$ is a continuous functional at every $x(s)$ of $C_1$.

A functional $F(x)$ defined on a neighborhood of $x_0(s)$ in $C_1$ to $C_1$ is said to have a *Fréchet differential* at $x(s) = x_0(s)$, if there exists a linear functional $F(x_0;y)$ of $y(s)$, depending parametrically on $x_0(s)$, that is a first order approximation to the difference $F(x_0 + y) - F(x_0)$ in the sense that $\eta(y)$ defined by

$$\eta(y) = F(x_0 + y) - F(x_0) - F(x_0;y)$$

has the property that given an $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $||\eta(y)|| \leq \epsilon||y||$ for all $y$ for which $||y|| < \delta(\epsilon)$. The linear functional $F(x_0;y)$ with the above property is called the *Fréchet differential of the functional $F(x)$ at $x(s) = x_0(s)$ of $C_1$ with increment $y(s)$ of $C_1$*. For emotional reasons we can also use the notation $\delta F(x_0;\delta x)$ in the place of $F(x_0;y)$ whenever the independent increment variable $y$ is written as $\delta x$.

One of the fundamental properties of a Fréchet differential of a functional is its unicity. One way of showing this is to show that a Fréchet differential can be evaluated as a limit. In fact it can be shown that the Fréchet differential $F(x_0;y)$ if it exists implies the existence of $\lim_{\lambda \to 0} \dfrac{F(x_0 + \lambda y) - F(x_0)}{\lambda}$ and its equality to $F(x_0;y)$. The limit symbolism $\lim_{\lambda\to 0}$ is to be interpreted in the following sense for the given $x_0(s)$ and for any chosen $y(s)$ of $C_1$: given an $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $\left\| \frac{F(x_0 + \lambda y) - F(x_0)}{\lambda} - F(x_0; y) \right\| < \epsilon$ for $0 < |\lambda| < \delta(\epsilon)$.

The mere existence of the limit $\lim_{\lambda\to 0} \frac{F(x_0 + \lambda y) - F(x_0)}{\lambda}$, called Gâteaux differential, even if it were a linear functional of $y(s)$, does not necessarily entail the existence of a Fréchet differential $F(x_0; y)$. however, if the limit $\lim_{\lambda\to 0} \frac{F(x_0 + \lambda y) - F(x_0)}{\lambda}$ exists for each $y(s)\epsilon C_1$, is a linear functional of $y(s)$, *and if* the limit is approached *uniformly* in $y(s)$ in some "hypersphere" $||y|| < a$, then the Fréchet differential $F(x_0; y)$ exists equal to this limit. This result was proved by the writer in the more general setting of the so-called *Banach spaces* of modern functional analysis.

A simple example of a Fréchet differential of a functional is no doubt instructive. Let the functional $F(x)$ on the above class $C_1$ of continuous functions be $F(x) = x(s)\int_a^b K(s,s_1)x(s_1)ds_1$. A direct computation yields

$$F(x_0 + y) - F(x_0) = y(s)\int_a^b K(s,s_1)x_0(s_1)ds_1 + x_0(s)\int_a^b K(s,s_1)y(s_1)ds_1$$
$$+ y(s)\int_a^b K(s,s_1)y(s_1)ds_1$$

Hence the Fréchet differential $F(x_0; y)$ exists equal to $y(s)\int_a^b K(s,s_1)x_0(s_1)ds_1 + x_0(s)\int_a^b K(s,s_1)y(s_1)ds_1$, for, $F(x_0 + y) - F(x_0) - F(x_0; y)$ is clearly given by $y(s)\int_a^b K(s,s_1)y(s_1)ds_1$ with the requisite property

$$\max_{a\le s\le b} \left| y(s)\int_a^b K(s,s_1)y(s_1)ds_1\right| \le \epsilon \max_{a\le s\le b} |y(s)| \text{ for } \max_{a\le s\le b} |y(s)| < \delta(\epsilon).$$

The differential calculus of functionals is a large subject. Although much important work has been done along this direction during the last two decades, the subject is still in its infancy and awaits a further grand development - both on its own account as well as for the possible applications to geometrv, group theory, various domains of classical analysis, and the Sciences.

There are unusual possibilities in the various new types of differential equations in which the *unknowns are functionals*. We shall give a simple existence and uniqueness theorem for a simple differential equation in functionals that occurs in the Fredholm theory of integral equations. Let $S$ be the set of all continnous kernels $K(x,s)$ of a

Fredholm integral equation (13) with non-vanishing Fredholm determinant $D[K]$. Let $k[K|x,s]$ be the resolvent kernel of $K(x,s)$. We saw in (17) that it is a functional of $K(x,s)$. If $\bar{y}[K|x]$ is a functional on $S$ to the space of the real continuous functions of $x$, $a \leq x \leq b$, and if $\delta\bar{y}[K|x]$ stands for the Fréchet differential of $\bar{y}[K|x]$ with arbitrary continuous increment $\delta K(x,s)$, we can write down the following differential equation

$$(*) \qquad \delta\bar{y}[K|x] = \int_a^b \bar{y}[K|s]\{\delta K(x,s) + \int_a^b \delta K(x,t)k[K|t,s]\,dt\}ds$$

in which the unknown is the functional $\bar{y}[K|x]$ whose independent variable $K(x,s)$ ranges over the set $S$. It can be shown that there exists a unique solution of the equation (*) that takes on the arbitrarily given initial condition

$$(**) \qquad \bar{y}[o|x] = y(x) \qquad [\, y(x) \text{ continuous in } (a,b)\,]$$

for the identically vanishing kernel $K(x,s)$. This unique solution of the differential equation (*) subject to the initial condition (**) is in fact

$$(***) \qquad \bar{y}[K|x] = y(x) + \int_a^b K(x\ s)y(s)ds$$

for all continuous kernels $K(x,s)$ with non-vanishing Fredholm determinant. Anyone who happens to be acquainted with the modern developments in continuous transformation groups will recognize the differential equation (*) in Fréchet differentials as the generalized Lie differential equations that characterize the Fredholm transformation group (***). This extremely simple and important transformation group is outside the pale of the Lie theories in that both the space over which the transformation group operates and the space of the parameters $K(x,s)$ are *not* finite dimensional spaces.

Although the pioneer work of Volterra on the theory of functionals began in 1887 and the need for existence and uniqueness theorems for differential equations with functionals as unknowns was outstanding from the very beginning, it was not until a half century later that satisfactory existence and uniqueness theorems were first given (see Acta Mathematica, 1937 where the subject is treated in the modern setting of abstract spaces with the differential equations in some function spaces considered as special cases).

Analytic functionals - including polynomials - of one form or another have been studied on many occasions since 1887. However, the modern work on the subject began in the year 1931-1932. For some of the work on special functionals and their characterization by differential equations in Fréchet differentials, the reader is referred to Acta Mathematica, 1948..

We have occasionally spoken of some modern work in the theory of functionals and its development within the framework of the theory of "abstract spaces." We shall attempt to clarify these relationships very briefly. The functions of many classes of functions - such as those discussed earlier - can be added for each value of the independent variable and thus produce another function of the same class. Furthermore, the functions of the class can often be multiplied by arbitrary numbers (say, real numbers) with the result that we have a new function of the same class. In other words, the class of functions forms a *linear space*. Furthermore, in a large "majority" of such classes there exists a generalized "absolute value", called a *norm* of the function. In terms of this norm, the notions of generalized analysis can be defined and developed (continuity, differentiability, convergence, limit processes, etc.) For *example*, if the class of functions is that of all real continuous functions $y(x)$ of a real variable $x$ for $a \leq x \leq b$, then the norm $||y||$ of $y(x)$ can be defined as

$$||y|| = \max_{a \leq x \leq b} |y(x)|.$$

Clearly

(*a*) $||y|| = 0$ if and only if $y(x) \equiv 0$ in $(a,b)$;

(*b*) $||\alpha y|| = |\alpha| \cdot ||y||$ for all real $\alpha$ and continuous $y(x)$;

(*c*) $||y_1 + y_2|| \leq ||y_1|| + ||y_2||$ (triangular inequality).

Furthermore, the generalized Cauchy convergence criterion holds since convergence (in the norm) means uniform convergence of continuous functions over a closed interval.

A class of elements that is a linear space and over which there exists a norm with the above properties (*a*), (*b*) and (*c*) such that the generalized Cauchy criterion for convergence holds is called a *Banach space*. Many of the classes of functions that are studied in the theory of functionals can be made into Banach spaces. It is for this reason that the "abstract" theory of functions in Banach space variables has an intimate connection with the theory of functionals as we have been discussing it. The modern theory of functionals has a large number of important and vast chapters while the applications of the theory encompass practically every nook and corner of modern mathematics. Some indications of this situation are given in Hille's "Functional Analysis and Semi-Groups," 1948 and in the author's "Functional Analysis in Topological Group Spaces," Mathematics Magazine, vol. XXI (1947), pp. 80-90.

Perhaps a simple application of the theory of functionals to some non-linear integral equations would be instructive at this point. If we use the same notations that we used in the definition of linear functionals and Fréchet differentials of functionals on a subset of $C_1$ to $C_1$ ($C_1$ is the Banach space of the real continuous functions of a real variable over a fixed, closed finite interval $(a,b)$ ), then we can readily define a homogeneous polynomial functional and an analytic functional. $P_n(x)$ is a homogeneous polynomial functional of degree $n$ on $C_1$ to $C_1$ if $P_n(x) = \Pi_n(x,x,\cdots,x)$ for some completely symmetric multilinear functional $\Pi_n(x_1,x_2,\cdots,x_n)$, *i.e.*, $\Pi_n(x_1,x_2,\cdots,x_n)$ is a linear functional (on $C_1$ to $C_1$) of each of the $n$ variables and is completely symmetric in the $n$ variables.

It can be shown that there exists a least positive $M$ (called the modulus of $P_n(x)$) such that $||P_n(x)|| \leq M\,||x||^n$. Let $P_n(x)$ be a homogeneous polynomial functional of degree $n$ with $M_n$ as modulus. Consider an infinite sequence $\{P_n(x)\}$ such that the real power series $\sum_n M_n\lambda^n$ has a radius of convergence $r > 0$. Hence $\sum_n P_n(x-x_0)$ converges for $||x-x_0|| < r$ and represents a functional $f(x)$ for $||x-x_0|| < r$. We shall call $f(x)$ an analytic functional of $x(s)$ at $x(s) = x_0(s)$ with a radius of convergence $r > 0$. It can be shown that successive Fréchet differentials of all orders of an analytic functional exist throughout the sphere of convergence $||x-x_0|| < r$, and they are computed by taking the Fréchet differentials of the expansion term by term.

Consider the following non-linear integral equation

$$(t) \qquad f = P_1(x) + \sum_{n=2}^{\infty} P_n(x)$$

where the known function $f(s)$ is in the class $C_1$ and the polynomial functionals $P_1(x)$, $P_2(x),\cdots$ are given. For the purposes of convenience in this exposition, we shall assume that $P_1(x)$ is the following linear functional on $C_1$ to $C_1$

$$x(s) + \int_a^b K(s,\ s_1)x(s_1)ds_1$$

in which $K(s,s_1)$ is a given continuous function of $s$ and $s_1$, in the square $a \leq s,\ s_1 \leq b$ with non-vanishing Fredholm determinant. We shall assume that $\sum_{n=2}^{\infty} P_n(x)$ is an analytic functional at $x(s) \equiv 0$ in $(a,b)$ with a radius of convergence $r > 0$. Under these restrictions, the non-linear integral equation $(t)$ has a unique analytic solution

$$(tt) \qquad x = \overline{P}_1(f) + \sum_{n=2}^{\infty} \overline{P}_n(f)$$

within the class of functions for which $||x|| < r_1$ and $||f|| < r_2$, where $r_1$ and $r_2$ are sufficiently small positive numbers. This solution $(tt)$ as a functional of $f(s)$ is an analytic functional of $f(s)$ at $f(s) \equiv 0$ in $(a,b)$ with a radius of convergence $r_2$. Other more general theorems can be proved.

Finally, to illustrate how the modern theory of functionals is instrumental in solving some outstanding problems in classical analysis, consider the system of $n$ ordinary numerical differential equations in the $n$ unknowns $x_1(s), x_2(s), \cdots, x_n(s)$

$$\frac{dx_i(s)}{ds} = \sum_{j=1}^{n} a_{ij}(s)x_j(s)$$

Assume that the given $n^2$ coefficients $a_{ij}(s)$ are continuous in the real closed finite interval $(a,b)$ and that the arbitrarily given initial conditions are $x_i(a) = x_{io}$ . It is possible to form Banach ppaces of functions in such a manner that the solutions of the differential equations subject to the given initial conditions are entire analytic functionals (radius of convergence $r$ is infinite) of the $n^2$ coefficients $a_{ij}(s)$ at any chosen $a^o_{ij}(s)$, continuous in $(a,b)$.

It is practically impossible to give here any adequate idea of the fundamental role that integral equations and the theory of functionals play in modern differential geometry (finite dimensional, infinite dimensional, and "dimensionless" ). We shall, in conclusion, signalize this subject without further remark and refer the interested reader to an American Mathematical Society invited lecture by the author entitled " General Differential Geometries and Related Topies.*"

* Bull. Amer. Math. Society, 1939.

California Institute of Technology
Pasadena, California

# CURRENT PAPERS AND BOOKS

*Edited by*

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

*Mathematics of Finance.* By Clifford Bell and L. J. Adams. Henry Holt and Co. New York, 1949, VII + 360 pgs. $2.75.

In eleven chapters this text covers the topics of simple interest and compound discount, annuities, installment buying, depreciation, bonds, permutations, combinations and probability, life annuities and life insurance.

Eighty seven pages in the back of the book are devoted to the tables usually found in texts on the mathematics of finance. Some of these tables give as many as eleven significant digits. These tables are well arranged and adequate. Answers are provided for all odd-numbered problems.

The mathematics used in the text is not beyond the ability of a good high school senior who has had the usual high school mathematics, however a pupil who has had a course in commercial algebra will be better prepared for the special commercial symbols used.

This book is "designed primarily for use in a three hour one semester course in mathematics of finance for college students majoring in commerce and business administration."

It would also be a good text to be used in a teachers college training teachers of mathematics.

C. N. Shuster

---

*Commercial Algebra.* By Clifford Bell and L. J. Adams. Henry Holt and Company. New York. 1949. VII + 304 pages.

This text was "designed for students who have had one year of algebra and is intended to furnish the mathematical background for business administration majors whose curriculum will include courses in accounting and mathematics of finance."

The text contains nearly all of the material found in the average second year algebra, but the subject matter has a decided commercial slant. It will adequately prepare a student for work in accounting and mathematics of finance and should be considered by any teacher who desires a good text in commercial algebra. The text would also by fully as satisfactory for regular (non-commercial) Algebra II classes.

While the text contains less of the relatively useless material found in so many algebra II texts, it does contain considerable material that would be very difficult to justify from the standpoint of any possible commercial use. Examples of the material cited above are: Find the algebraic sum:

(1) $5\sqrt{\frac{3}{8}} - \sqrt{24} + 4\sqrt{\frac{2}{3}}$

(2) Divide $\sqrt{3} + \sqrt{5}$ by $\sqrt{3} - \sqrt{5}$

(3) $\dfrac{a^2 - 9a + 14}{a^2 - 5a + 6} \div \dfrac{a^2 - 10a + 21}{a^2 - 7a + 12}$

The text has a very satisfactory six-place table of logarithms, several other tables and answers to all odd problems.

C. N. Shuster

---

*The geometry of the zeros of a polynomial in a complex variable.* By Morris Marden. *Mathematical Surveys,* Number III. American Mathematical Society, 531 W. 116th Street, New York, New York, 1949, ix + 183 pp. $5.00.

The analytic theory of equations, as distinct from the algebraic theory, dates from the introduction of the Argand diagram into mathematics, Gauss and Cauchy having made fundamentally important contributions to the subject. Since their pioneer work, a very extensive literature has developed, the bibliography in the book under review listing 480 separate publications in the field, exclusive of such no doubt purposely omitted references as the relevant sections in the standard treatises on analysis, *e.g.,* Goursat's *Cours d'Analyse.* In the last half century several partial surveys of this vast literature have appeared, culminating in Dieudonné's recent comprehensive monograph in the *Mémorial des Sciences Mathématiques* series. However, even Dieudonné's excellent book, because of space limitations, is definitely restricted in scope; and, consequently, there has remained a real need for a detailed exposition of the whole subject, bringing together and applying results widely scattered throughout the literature of mathematics and unifying and simplifying both these results and the methods by which they have been obtained. Marden's book, the third volume in the *Mathematical Survey* Series, was designed to fulfill the need just alluded to and succeeds admirably in its avowed purpose.

The material in Marden's book subdivides naturally into a first part

(Chapters I to VI) concerned with the study of the zeros $Z_1$, $Z_2$, $\cdots$ $Z_m$ of a polynomial $F(z)$ regarded as functions $Z_k = Z_k(z_1, z_2, \cdots z_n)$ of some or all of the zeros $z_k$ of a *related* polynomial $f(z)$; and a second part (Chapters VII to X) devoted to the investigation of the zeros $z_k$ of a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ regarded as functions $z_k = z_k(a_0, a_1, \cdots, a_n)$ of some or all of the coefficients $a_j$ of $f(z)$. The problems treated in the second part of the book fall chiefly into two groups: a first category in which given an integer $p$, $1 \leq p \leq n$, we seek to find a region $R = R(a_0, a_1, \cdots, a_n)$ containing at least or exactly $p$ zeros of the polynomial $f(z)$; and a second category in which given a region $R$ we seek to find the number $p = p(a_0, a_1, \cdots, a_n)$ of zeros of $f(z)$ in $R$.

A number of the results obtained in the first part of the book will be recognized by the reader as complex variable analogs of Rolle's familiar theorem. Similarly, certain of the results secured in the second part are complex variable analogs of the classical rules of sign of Descartes and Sturm.

Chapter I is devoted to consideration of such basic theorems as Cauchy's Principle of Argument; and to dynamical, hydrodynamical, geometric and function-theoretic interpretations of the critical points of polynomials in the complex variable $z$. Chapter II continues the study of such critical points and of some of their generalizations, presenting such important theorems as those of Lucas and Jensen and related theorems due to Walsh and Marden. Chapter III is devoted to generalizations of results obtained in the first two chapters, the method employed being that of conformal mapping and a particularly detailed treatment being given of Laguerre's important theorem. Chapter IV introduces Grace's theorem and in this and the next chapter, many important applications are made of this interesting and useful theorem and of cognate theorems due to Walsh and Marden in the study of problems relating to the zeros of linear combinations of polynomials. In chapter VI the extent to which prescription of some of the zeros of $f(z)$ determines the location of some of the critical points of $f(z)$ is investigated. In Chapter VII Marden secures bounds, valid either for all or for some of the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, as functions of all of the coefficients $a_j$, $j = 1, 2, \cdots, n$. In contrast, in Chapter VIII, Marden obtains, for the $p < n$ zeros of $f(z)$ of smallest modulus, bounds which are independent of certain of the coefficients $a_j$. Chapter IX attacks the dynamically important problem of finding exactly or approximately the number of zeros of a polynomial $f(z)$ lying in a prescribed region, *e.g.*, a half-plane or sector. Finally, Chapter X is devoted to a detailed study of the problem of determining the number of zeros of $f(z)$ lying in a prescribed region in the special case in which this region is a circle.

It is evident that the book under review has been written, edited, and printed with meticulous care. Such typographical slips as the substitution of $m_1$ for $n_1$ in one term of the formula for $D$ on page 80; and of $n_2$ for $m_1$ in an exponent in formula (38.12) on page 132 will occasion the reader no difficulty and more serious errors are notable by their absence. For consistency's sake, it would have been well to include the abbreviation "deg $Q_k(x)$" in the list of abbreviations on page ix; and the problem of dynamic stability in Chapter IX could have been introduced more simply by reference to the notion of equations of variation. However, the quite minor nature of these strictures only serves to emphasize the general excellence of Marden's book, which sets the highest standards before future authors in the *Mathematical Surveys* Series.

University of New Mexico Lincoln La Paz

---

*Partial Differential Equations in Physics.* By Arnold Sommerfeld. (Translation by Ernst G. Straus of Sommerfeld's *Vorlesungen über theoretische Physik,* Band VI—*Partielle Differentialgleichungen der Physik*—Akademischen Verlagsgesellschaft Geest and Portig K. G. in Leipzig, Sept. 1947.) Academic Press Inc., Publishers, New York, N.Y., 1949, xi + 335 pp.

The technique of applying mathematics to the manifold problems encountered by the physicist and the engineer may be taught by mathematicians in two ways: On the one hand, selected branches of mathematics may be systematically developed, the theory receiving illustration by means of treatment of judiciously chosen problems relating to the physical world. On the other hand, a choice may be made of representative problems, consideration of which serves to reveal the physical motivation of those mathematical methods which have been found not only to suffice for the solution of the specific examples chosen (and indeed for entire classes of analogous problems) but also to give rise to comprehensive, esthetically pleasing, mathematical theories promising almost unlimited flexibility and usefulness. A case in point is the voluminous and beautiful theory of the representation of arbitrary functions in infinite series, an outgrowth of the method Fourier devised for the solution of a certain type of physical problem. That just such all-inclusive, logically satisfying theories chance to be of paramount interest to the mathematician is not the only, or even the principal, reason for teaching them to students of physics and engineering. After all in the rapidly changing world of modern science the alert applied mathematician can afford to be no more interested in the restricted view provided by *solved* problems than in the wider picture afforded by general theories which give promise of sufficient flexibility and power to overcome as yet *unsolved* or even *unformulated* problems.

The book under review is an outstandingly successful example of the second way of teaching applied mathematics, the problems chosen to reveal the motivation of the mathematics employed ranging from classical questions formulated by LaPlace and Fourier to ultra-modern topics in the field of radio-wave propagation.

In Chapter I Fourier series are introduced in connection with best approximation, in the least squares sense, to an arbitrary function by means of trigonometric polynomials,

$$S_n(x) = A_0 + A_1 \cos x + \cdots + A_n \cos nx + B_1 \sin x + \cdots + B_n \sin nx .$$

Sommerfeld stresses that since the values found in the usual way for the Fourier coefficients $A_k$, $B_k$, $k < n$, are independent of $n$, these coefficients remain unchanged when we pass from $n$ to $n + 1$; and, hence, may be said to satisfy a *condition of finality*. Not only is this finality condition a consequence of the classical orthogonality condition but, conversely, orthogonality can, in general, be deduced from the requirement of finality. This fact permits the more formal computations of the older developments to be supplanted by a complete and generalizable procedure not only in the trigonometric case, but also for spherical harmonics and general eigenfunctions.

Chapter II, an introduction to partial differential equations, deals almost exclusively with linear partial differential equations of the second-order in two independent variables, such equations being classified as elliptic, parabolic or hyperbolic on the basis of the nature of the associated families of Monge-characteristics. The quite different boundary conditions relevant to each of the three types of equation are carefully pointed out; differential expressions adjoint and self-adjoint in the sense of Frobenius are defined; and, after a treatment of considerable generality of Green's theorem and Green's function, the various representations of the solutions to which Green's transformation leads in the three cases are discussed. This chapter concludes with an interesting exposition of Riemann's ingenious method for solving equations of hyperbolic type and the application of Green's theorem to heat conduction problems. The latter topic serves to introduce Chapter III, in which the classical problems of heat conduction are attacked not only by Fourier's method but also by use of Green's functions constructed by the method of reflected images for regions with plane boundaries.

Sommerfeld seems somewhat apologetic about the length of Chapter IV— together with its appendices it makes up nearly one-fourth of his entire volume. However, the reader can only marvel at the wealth of material on Bessel, Hankel, Green and hypergeometric functions; on spherical harmonics; and on inversion and potential theory, all of which have been thoroughly digested and then presented in elegant and usable form in the relatively brief space of this fourth chapter.

In Chapter V on eigenfunctions and eigen values, methods of great generality, stemming from Fourier's fundamental notions, are shown to develop quite naturally from the viewpoint of partial differential equations. This chapter can scarcely fail to impress the reader at once with the power of the methods employed and with Sommerfeld's extraordinary skill in *adsorbing*, to borrow a physico-chemical term, an entire atmosphere of applications on each grain of truth ground out of the mathematical mill. However, the reviewer confesses surprise that this otherwise excellent chapter contains no reference to the powerful and convenient variational methods developed by Rayleigh, Ritz and their followers. The interesting connection between the Rayleigh-Ritz and the eigenfunction-method employed by Sommerfeld has been known at least since J. Tamarkin's paper of 1918 on the method of Ritz.

The last chapter in the book under review is devoted to what a communications engineer familiar with the terminology of ballistic theory on a flat, airless non-rotating earth might be tempted to describe as the academic problem of radio-wave propagation. Attention is restricted to dipole-antennas (in a footnote mention is made of antennas of *finite* length but the reference is to a 1942 paper by A. Sommerfeld and F. Renner criticised from the viewpoint of communications engineering by M. C. Gray in 1943); the reflection processes in the ionosphere of such fundamental importance in the propagation of actual radio waves and questions relating to the construction of practical transmitters and receivers are not treated; and, in contrast to the earlier chapters, numerical details are notable only by their absence. Nevertheless, this final chapter is exceedingly heuristic and should be read with interest by engineer and mathematician alike.

A list of exercises to which have been relegated proofs of a number of theorems which Sommerfeld did not have space to treat; a section containing hints as to how to solve these exercises (wisely set off by itself); and an index so little detailed that it lacks, for example, an entry relative to the important condition of finality complete the book under review.

There remains appraisal of the skill and care with which the book has been written, translated, edited and printed. Having regard to the typographical difficulties involved, the printer's work is satisfactory except for occasional illegibilities and blemishes (p. 61, down 12; p. 153, down 3; p. 304, (9b); and p. 326, up 7 are cases in point). As regards the translation, the English reader will probably regret that "of" is twice omitted in the last paragraph of §16; and will certainly wish to revise part of the caption under fig. 7, p. 28 to read, "The graphs of both $y = \tan \lambda\pi$ and $y = \alpha\lambda$ have been drawn on the same set of axes." Furthermore, a curious mistake in translation occurs in the third sentence on p. 201 where the genetive singular, ***Planeten***, is mistaken for the plural form. (Since "Kepler's problem", referred to by Sommerfeld, is a two-body problem, the sense of the original German was certainly that ". . . the nucleus plays the role of the sun and the electron that of the planet.") Nevertheless, the translator, almost without exception, has done an excellent job. A few errors have escaped both Sommerfeld's scrutiny and editorial watchfulness and have been carried over from the German to the English version. These errors and certain additional corrections are enumerated in the list that follows:

On p. 5, up 13, read §5 instead of §4; on p. 9, in fig. 2, the symbols $S_1$, $S_3$, $S_5$ should follow the word "approximations"; on p. 35, up 7, read (5) instead of (15); on p. 61, the last term in (18) arises from decomposition of the second integral in (17) into the *difference* of the components corresponding to the rod ends, $x_0$ and $x_1$; on p. 74, §16, paragraph 1, the last sentence should read "The meaning of $\tau$, $\cdots$ , must then be amended in one or the other of the ways described on the following page"; p. 75, *a*)*a*) and *b*)*b*) should be supplemented by inserting the definition $\tau = \pi ikt/l^2$ and by replacing the symbol $G$ by $2lG$ while, on the contrary, *a*)*b*) and *b*)*a*) should be supplemented by inserting the definition $\tau = \pi ikt/4l^2$ and by replacing the symbol $G$ by $4lG$; p. 83, (7) read $u_1 = u_2$ instead of $u_1 = u$; p. 101, fig. 21, on the axis of ordinates transfer the parenthesis from $I_0$ to $I_1$; p. 154, the translation omits the following footnote which, in the German version, is appended to the term "confluent" in the first line of the last paragraph: "The Germanism *entartet* is not to be employed in this connection, since this word is used in another sense, particularly in wave mechanics. In any event, one should not depart from international usage!" (one who turns from this admonition to the Bessel function notation advocated by Sommerfeld on p. 87 will find that the author has not heeded his own advice.)

In conclusion, on balancing out the few quite minor imperfections of the book under review against its many excellences, the reader will not hesitate to accord hearty commendation to author, translator and editors alike for tasks exceedingly well done.

University of New Mexico Lincoln La Paz

---

*Arithmetic for Colleges.* By Harold D. Larsen. New York. The Macmillan Company, 1950. 11 + 275 pages. $3.75.

According to the author, this book was designed for a one semester course in the principles and applications of elementary arithmetic for use in elementary teacher-training classes.

The usual material on addition, subtraction, multiplication, division, common fractions, decimal fractions, percentage, and denominate numbers is well-covered. An abundance of material of an historical nature is scattered throughout the text which, added to the treatment of short cuts, approximate numbers, and the slide rule, gives the reader many interesting topics of arithmetic. It is unfortunate that practically all the material on commercial arithmetic has been omitted and in addition the author gives little or nothing on methods of teaching; in fact he appears to be concerned mainly in increasing and widening the prospective teacher's knowledge in arithmetic rather than in telling him how to teach.

It is the reviewer's opinion that this book would make an excellent text for a review course in arithmetic for those who feel that they need a better understanding of the fundamentals of arithmetic. It would also serve as an excellent text for an elementary teacher-training class provided that the instructor could supplement it with the proper material on teaching methods and techniques. Furthermore it is believed that all teachers will find this book to be a valuable source from which they can find material that will improve their teaching.

University of California
Los Angeles Campus

Clifford Bell

# PROBLEMS AND QUESTIONS

*Edited by*

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known text-books should not be submitted.

In order to facilitate their consideration, solutions should be submitted on separate, signed sheets within three months after publication of the problems. Readers are invited to offer heuristic discussions in addition to formal solutions.

All manuscripts should be typewritten on 8½″ by 11″ paper, double-spaced and with margins at least one inch wide. Figures should be drawn in india ink and in exact size for reproduction.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, Calif.*

## PROPOSALS

**77.** *Proposed by H. E. Bowie, American International College.*

A farmer has a given length of fencing. He wishes to construct a given number of rectangular pens by running two fences from East to West and the necessary number of fences from North to South. If the area of the pens is to be a maximum, what relation exists between the total length of the North-South fences and the total length of the East-West fences?

**78.** *Proposed by P. A. Piza, San Juan, Puerto Rico.*

$t_r = r(r+1)/2$ is a triangular number of order $r$. Find a general solution of the equation

$$t_x + t_y = z^2 + (y - z)^2.$$

**79.** *Proposed by Norman Anning, University of Michigan.*

Show, preferably by finding factors, that $A + B = 60°$ is a sufficient but not a necessary condition in order that we may have $\cos^2 A + \cos^2 B - \cos A \cos B = 0.75$.

**80.** *Proposed by R. E. Horton, Los Angeles City College.*

In tetrahedron $ABCD$ a plane is passed through vertex $D$ and parallel to edge $BC$ forming another tetrahedron $ARSD$. If the ratio of the volumes of these tetrahedra is a rational perfect square, then the plane cuts the median of $ABCD$ from $A$ into two parts whose ratio is a rational number; and conversely.

**81.** *Proposed by E. P. Starke, Rutgers University.*

There is reason to suppose that in the long run and with a very large number of pupils in a properly instructed and properly tested mathematics class there would be approximately 12½% who fail to attain a satisfactory grade. Because she wants to do everything in the most modern, scientific manner, Miss X adjusts her grades in each class of 24 pupils so that there are always exactly 3 failures (and the approved number of *A*'s, *B*'s, etc.)

Assuming that 12½% is the correct probability of a student's failing and that Miss X teaches and tests satisfactorily, show that, on the average, less than one class (24 pupils) in four should have exactly three failures each. How often should Miss X expect a class in which half or more of the pupils ought to fail?

**82.** *Proposed by W. R. Talbot, Jefferson City, Mo.*

Show that the abscissas of the roots, the maximum and minimum points, and the inflection points for a quartic function cannot be distinct integers.

**83.** *Proposed by Victor Thébault, Tennie, Sarthe, France.*

If the Lemoine point of a triangle *ABC* lies on the circumcircle of the tangential triangle of *ABC* we have $a^2 + b^2 + c^2 = 6R^2$ where $a$, $b$, $c$ are the sides and $R$ is the circumradius of *ABC*.

## SOLUTIONS

### A Property of the Complete Quadrilateral

**54.** [Jan. 1950] *Proposed by Howard Eves, Oregon State College.*

If one side of a given complete quadrilateral is parallel to the Euler line of the triangle formed by the remaining three sides, the same is true for every side of the quadrilateral.

*Solution by the Proposer.* We shall employ vector algebra. Let $u$, $v$, $w$ be position vectors of the vertices of a triangle and set

$$r = v - w, \quad s = w - u, \quad t = u - v,$$

$$D = (t\cdot r)(r\cdot s) + (r\cdot s)(s\cdot t) + (s\cdot t)(t\cdot r),$$

$$h = [(t\cdot r)(r\cdot s)u + (r\cdot s)(s\cdot t)v + (s\cdot t)(t\cdot r)w]/D.$$

Then

$$h - u = [(r\cdot s)(s\cdot t)(v - u) + (s\cdot t)(t\cdot r)(w - u)]/D, \qquad (1)$$

with similar expressions for $h - v$ and $h - w$. We now easily find that

$$(h - u)\cdot(v - w) = (h - v)\cdot(w - u) = (h - w)\cdot(u - v) = 0,$$

whence $h$ is the orthocenter of triangle $uvw$. Adding together the three expressions like (1) we find, where $g$ is the centroid of triangle $uvw$,

$$q \equiv D[3h - (u + v + w)] = 3D(h - g)$$
$$= r[(r\cdot s)(s\cdot t) - (s\cdot t)(t\cdot r)] + s[(s\cdot t)(t\cdot r) - (t\cdot r)(r\cdot s)]$$
$$+ t[(t\cdot r)(r\cdot s) - (r\cdot s)(s\cdot t)].$$

We now have, in $q$, a vector expression for the direction of the Euler line of triangle $uvw$. Using this expression and the fact that $r + s + t = 0$ we may verify, by tedious but straightforward substitution, that

$$(s\cdot t)(r\cdot q) + (t\cdot r)(s\cdot q) + (r\cdot s)(t\cdot q) = 0.$$

Since this relation is linear in each of $r$, $s$, $t$, $q$ we then also have

$$(s'\cdot t')(r'\cdot q') + (t'\cdot r')(s'\cdot q') + (r'\cdot s')(t'\cdot q') = 0,$$

where $r'$, $s'$, $t'$, $q'$ are any vectors parallel to $r$, $s$, $t$, $q$ respectively. Here, then, is a relation existing between four directions – the directions of the sides and of the Euler line of a triangle. But the relation is symmetric in these directions. Therefore any three of the directions may be taken as the directions of the sides of a triangle, and the fourth direction will be that of the Euler line of this triangle. This proves the theorem.

For a synthetic proof see Henry E. Fettis, The Complete Quadrilateral, *This Magazine*, **22**, 22, (Sept. 1948).

Also solved by W. B. Carver, Cornell University, (using conjugate coordinates); and P. D. Thomas, Washington, D. C., (using trigonometry).

## A Whirling Bucket

**55.** [Jan. 1950] *Proposed by R. E. Winger, Los Angeles City College.*

a) A cylindrical bucket of radius $r$ and height $h$ is $1/k$ full of liquid. Find the maximum speed at which the bucket may be whirled about its vertical axis of symmetry without losing any of the liquid. Neglect surface tension.

b) What is the maximum value of $k$ for which no part of the bottom of the bucket will be dry at the maximum speed.

*Solution by the Proposer.* Since a liquid cannot support a shearing stress, the resultant of the nonviscous internal forces acting on a surface particle is normal to the surface. When the system has been brought up to a constant angular speed $\omega$, the resultant of the internal forces and the gravitational force $mg$ acting on a particle at distance $x$ from the axis of rotation is the unbalanced centripetal force $mx\omega^2$. Hence the slope of the intersection of the surface of revolution with a plane through the axis is $dy/dx = x\omega^2/g$. Therefore the equation of the intersection is $y = x^2\omega^2/2g + C$. If the liquid just reaches the rim of the bucket at speed $\hat{\omega}$, and the origin is chosen at the vertex of the paraboloid, then $y_r = r^2\hat{\omega}^2/2g$. The solution of the problem will be completed by treating two cases.

*Case I.* Vertex above the bottom of the bucket. The volume of the bucket devoid of liquid is

$$V = \int_0^{r^2\hat{\omega}^2/2g} \pi x^2 dy = \frac{2g\pi}{\hat{\omega}^2}\int_0^{r^2\hat{\omega}^2/2g} y\, dy = \pi r^4\hat{\omega}^2/4g.$$

Now $V$ is also equal to $\pi r^2 h(1 - 1/k)$. Equating the expressions for $V$ and solving, $\hat{\omega} = \frac{2}{r}\sqrt{gh(k - 1)/k}$ radians/sec. If no part of the bottom is to be dry, then $h \geq r^2\hat{\omega}^2/2g$, so $k \leq 2$.

*Case II.* Vertex below bottom of bucket.

$$V = \int_{r^2\hat{\omega}^2/2g-h}^{r^2\hat{\omega}^2/2g} \pi x^2 dy = \pi h\left[r^2 - \frac{gh}{\hat{\omega}^2}\right] = \pi r^2 h\frac{(k - 1)}{k}.$$

Solving,

$$\hat{\omega} = \sqrt{kgh}/r \text{ radians per second.}$$

b) As in Case I, if no part of the bottom is to be dry, the bottom is tangent to the paraboloid, so $h = r^2\hat{\omega}^2/2g$ and $k = 2$, which is the maximum value of $k$ for which no part of the bottom will be dry when the liquid touches the rim.

Also solved by Robert Kissling, Berkeley, California and W. I. Thompson, Los Angeles City College.

### A Special Square Number

**56.** [March 1950]. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Find a nine-digit integer of the form $a_1a_2a_3b_1b_2b_3a_1a_2a_3$ which is the product of the squares of four distinct primes, $a_1 \neq 0$, $b_1b_2b_3 = 2(a_1a_2a_3)$.

*Solution by P. N. Nagara, College of Agriculture, Thailand.* $N^2 = a_1a_2a_3b_1b_2b_3a_1a_2a_3 = (a_1a_2a_3)(1002001) = (a_1a_2a_3)(7^2)(11^2)(13^2)$. Therefore, $a_1a_2a_3$ must be the square of a prime, $P$, other than 7, 11, or 13. Now $a_1 \neq 0$ and $b_1b_2b_3 < 1000$, so $10 < P < 23$. Hence $P = 17$ or 19, $a_1a_2a_3 = 289$ or 361 and $N^2 = 289578289$ or 361722361.

Also solved by P. A. Hinrichs, Hofstra College; F. L. Miksa, Aurora, Ill.; Alfred Sylwester, Corvallis, Oregon; L. A. Ringenberg, Charleston, Ill.; W. R. Talbot, Jefferson City, Mo.; and the proposer.

### How Old was Henry the Tiler?

**57.** [March 1950] *Proposed by J. S. Cromelin, Clearing Industrial District, Chicago, Ill.*

The floor of a room twice as long as wide was laid by Art, Jake, Pete and Henry with 6″ by 6″ tile. Henry noticed that the length of the room in feet equalled his age. The boss, who had 36 dollar-bills and 40 dimes, paid off the men at the rate of two cents per tile. Art got \$2.00 more than Jake, Pete got 40 cents more than Art, and each was paid just what he earned. Henry was given \$5.00 less than Jake, but had to give back some change. All this happened on Henry's birthday. How old was he?

*Solution by M. Morduchow, Polytechnic Institute of Brooklyn, N.Y.* Let $A$, $J$, $P$, and $H$ be the number of tiles laid by Art, Jake, Pete and Henry respectively. Then $A = J + 100$, and $P = A + 20$. Moreover, it may be assumed (since the boss had sufficient dimes) that Henry gave back less than 10 cents change. It then follows that $H = J - 250 - x$, where $x$ is an integer between 1 and 4 inclusive. If $n$ is the number of tiles laid along the width, then $2n$ is the number laid along the length, and $2n^2$ is the total number of tiles laid. This can be expressed by the equation $A + P + J + H = 2n^2$. Substituting in this equation for $P$, $J$ and $H$ in terms of $A$, one obtains: $4A = 2n^2 + 430 + x$. Observing that $H = A - 350 - x$, and that $H$ must be positive, it follows that $A$ must exceed 351. Moreover, since the boss had only \$40, it follows that the total number of tiles laid did not exceed 2000. These conditions imply that $n$ must be between 23 and 31 inclusive.

Since the boss had only dollar bills and dimes, and only Henry had to give change, it follows that $A$ must be divisible by 5. Hence $4A$ must be divisible by 20. The problem is thus reduced to the following: Find the pair of integers $n$ and $x$, where $n$ is between 23 and 31, and $x$ between 1 and 4, such that the quantity $(2n^2 + 430 + x)$ is divisible by 20. Now $2n^2$ must terminate in 0, 2, or 8, so $x = 2$. Then the penultimate digit of $2n^2$ must be even, so $n = 28$. Therefore, 56 tiles were laid lengthwise, the length was 28 ft., so Henry was 28 years old.

The amounts paid were: Art – \$10.00, Jake – \$8.00, Pete – \$10.40, and Henry – \$2.96.

Also solved by F. L. Miksa, Aurora, Ill., L. A. Ringenberg, Eastern Illinois State College; and the proposer. One incorrect solution was received.

### A Locus Connected with the Parabola

**60.** [March 1950] *Proposed by Victor Thébault, Tennie, Sarthe, France*

The tangents $MA$, $MB$ from a point $M$ to a parabola $(P)$ meet the curve in the distinct points $A,B$. Find the locus of the point $M$, if the circle $MAB$ passes through the vertex of $(P)$.

*Solution by Howard Eves, Oregon State College.* Let us use cartesian coordinates, taking $y^2 = kx$ as the equation of $(P)$. Let the coordinates of $M$, $A$, $B$ be $(m,n)$, $(a,b)$, $(c,d)$, respectively. Since

$$(b - n)/(a - m) = \text{slope at } A = k/2b,$$

we readily find that we may take

$$b = n + \sqrt{n^2 - km}, \qquad d = n - \sqrt{n^2 - km}.$$

Now the condition that the circle *MAB* pass through the origin [which is the vertex of (*P*)] is that

$$\begin{vmatrix} m^2+n^2 & m & n \\ a^2+b^2 & a & b \\ c^2+d^2 & c & d \end{vmatrix} = 0.$$

Now

$$\begin{vmatrix} m^2+n^2 & m & n \\ a^2+b^2 & a & b \\ c^2+d^2 & c & d \end{vmatrix} = \begin{vmatrix} m^2+n^2 & m & n \\ b^4/k^2+b^2 & b^2/k & b \\ d^4/k^2+d^2 & d^2/k & d \end{vmatrix}$$

$$= \begin{vmatrix} m^2+n^2-km & m & n \\ b^4/k^2 & b^2/k & b \\ d^4/k^2 & d^2/k & d \end{vmatrix} = \frac{bd(b-d)}{k^4} \begin{vmatrix} m^2+n^2-km & m & n \\ b^2+d^2+bd & k & 0 \\ d^3 & kd & k^2 \end{vmatrix}$$

$$= \frac{bd(b-d)}{k^3}\{nbd(b+d) + k^2(m^2+n^2-km) - km[(b+d)^2 - bd]\}. \quad (1)$$

But

$$bd = km, \qquad b + d = 2n, \qquad b - d = 2\sqrt{n^2 - km}.$$

Substituting these in (1) we get $2m(n^2 - km)^{3/2}(k - 2m) = 0$. Thus the required locus consists of the straight lines $x = 0$ and $x = k/2$. We disregard the locus $y^2 - kx = 0$, for if $M$ is on (*P*) the triangle *MAB* degenerates into a point. Restricting ourselves to only the situations where the triangle *MAB* is real and non-vanishing we may then describe the required locus as consisting of parts of two parallel straight lines: the tangent to (*P*) at its vertex and the line parallel to this tangent such that the focus of (*P*) lies midway between the two lines. All portions of these lines which lie on or inside (*P*) must be omitted from the locus.

Also solved by H. E. Fettis, Dayton, Ohio; L. M. Kelly, Michigan State College; P. N. Nagara, College of Agriculture, Thailand; L. A. Ringenberg, Charleston, Ill.; and P. D. Thomas, Washington, D.C.

## A Direct Product of Two Determinants

**61.** [March 1950] *Proposed by R. E. Horton, Los Angeles City College.*

Find the value of the determinant $D$ in which $a = \sin\theta$ and $b = \cos\theta$.

$$D = \begin{vmatrix} a^3 & a^2b & ab & b^2 & ab & b^2 \\ a^2b & a^3 & b^2 & ab & b^2 & ab \\ ab & b^2 & a & b & a & b \\ b^2 & ab & b & a & b & a \\ ab & b^2 & a & b & 0 & 0 \\ b^2 & ab & b & a & 0 & 0 \end{vmatrix}$$

*I. Solution by the Proposer.* Let $\begin{vmatrix} a & b \\ b & a \end{vmatrix} = P.$

Partitioning out two by two sub-determinants in $D$ we have

$$D = \begin{vmatrix} a^2\cdot P & b\cdot P & b\cdot P \\ b\cdot P & 1\cdot P & 1\cdot P \\ b\cdot P & 1\cdot P & 0\cdot P \end{vmatrix} = P \times \begin{vmatrix} a & b & b \\ b & 1 & 1 \\ b & 1 & 0 \end{vmatrix}.$$

This direct product of two determinants, $P \times Q = P^3Q^2$. [See, for example, C. C. MacDuffee, *An Introduction to Abstract Algebra,* (1940), page 248.] Hence $D = (a^2 - b^2)^3(b^2 - a^2)^2 = -(b^2 - a^2)^5 = -(\cos^2\theta - \sin^2\theta)^5 = -\cos^5 2\theta$.

*II. Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota.* We perform the operations $\text{col}_1 - b\,\text{col}_3$, $\text{col}_2 - b\,\text{col}_4$, $\text{col}_3 - \text{col}_5$, $\text{col}_4 - \text{col}_6$, $\text{col}_5 - \text{col}_6$ and take common factors from the columns. Next, we first perform the operations $\text{row}_2 + \text{row}_1$, $\text{row}_3 + \text{row}_4$, $\text{row}_6 + \text{row}_5$, and then $\text{col}_1 - \text{col}_2$, $\text{col}_3 - \text{col}_4$, $\text{col}_6 - b\,\text{col}_2$, and take common factors from the rows and columns. .Thus we have,

$$D = (a^2-b^2)^2(a-b)\begin{vmatrix} a & b & 0 & 0 & b & b^2 \\ b & a & 0 & 0 & -b & ab \\ 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & -1 & a \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \end{vmatrix} = (a^2-b^2)^5 \begin{vmatrix} 1 & b & 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & a \\ 0 & 0 & 1 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

Obviously, the last determinant equals +1. Therefore $D = (a^2 - b^2)^5 = -(b^2 - a^2)^5 = -(\cos^2\theta - \sin^2\theta)^5 = -\cos^5 2\theta$.

Also solved by F. J. Duarte, Caracas, Venezuela; W. H. Glenn, Jr., John Muir College, Pasadena, Calif.; Aida Kalish, Polytechnic Institute of Brooklyn, N. Y.; F. L. Miksa, Aurora, Ill.; M. Morduchow, Polytechnic Institute of Brooklyn, N. Y.; P. N. Nagara, College of Agriculture, Thailand; L. A. Ringenberg, Charleston, Ill.; W. R. Talbot, Jefferson City, Mo.; and the proposer (in a second way). One incorrect solution was received.

**An Elliptical Cylinder on An Inclined Plane**

**62.** [March 1950]. *Proposed by E. P. Starke, Rutgers University.*

A right cylinder whose cross-section is an ellipse of eccentricity $e$ is placed with its axis horizontal upon an inclined plane, and it does not roll down. What is the greatest possible inclination of the plane? Assume there is no resistance to rolling but that the cylinder cannot slide down the plane.

*I. Solution by H. E. Fettis, Dayton, Ohio.* From the geometry of the configuration it is evident that the cylinder will be in equilibrium if the center of gravity of an elliptic section is directly above its point of contact with the inclined plane. The problem is then to determine the angles of inclination for which such a position of the cylinder is possible.

Assuming the cylinder to be in its equilibrium position, so that the diameter through the point of contact is vertical, let the half-length of this diameter be $b'$, and that of the conjugate diameter (obviously parallel to the plane) be $a'$. Then if $\theta$ is the angle of inclination of the plane, the angle between the diameters is $\pi/2 - \theta$. Also, $a$, $b$ being the semi-axes of the ellipse,

$$a'b' = ab \sec\theta, \quad a'^2 + b'^2 = a^2 + b^2. \tag{1}$$

[See G. Salmon, *Conic Sections*, (1929), pp. 168-170.] The parametric equations of the ellipse referred to its axes, in terms of the eccentric angle $\phi$, are $x = a \cos\phi$, $y = b \sin\phi$. The eccentric angle for the point of contact is then given by

$$b'^2 = a^2\cos^2\phi + b^2\sin^2\phi. \tag{2}$$

Elimination of $a'$ and $b'$ between (1) and (2) gives, upon simplification,

$$\sec^2\theta - 1 = [(a^2 - b^2)/2ab]^2\sin^2 2\phi = e^4\sin^2 2\phi/4(1 - e^2)$$

or

$$\tan\theta = e^2\sin 2\phi/2\sqrt{1 - e^2}.$$

Thus, for an equilibrium position to exist, we must have

$$\tan\theta \le e^2/2\sqrt{1 - e^2},$$

and the required greatest possible inclination is

$$\text{arc} \tan(e^2/2\sqrt{1 - e^2}) = \text{arc} \sin[e^2/(2 - e^2)].$$

*II. Solution by Howard Eves, Oregon State College.* Let us take the $x$-axis along and up the incline and the $y$-axis out from the incline along the minor axis of one of the elliptical cross-sections when the cylinder has its major axis parallel to the incline. The equation of the ellipse then becomes

$$x^2/a^2 + (y - b)^2/b^2 = 1,$$

where $a$ and $b$ are the semi-major and semi-minor axes respectively. If the ellipse is rolled along the $x$-axis, its center traces a curve. The differential equation satisfied by this roulette may be found in the standard way as follows. Let $r$ be the distance from the center of the ellipse to a point $(x_1, y_1)$ on the ellipse, and let $p$ be the perpendicular distance of the center from the tangent drawn at $(x_1, y_1)$. Then we readily find that

$$r^2 = x_1^2 + (y_1 - b)^2, \quad p^2 = [x_1^2(a^2 - b^2) - a^2r^2]^2/[a^4r^2 - x_1^2(a^4 - b^4)].$$

Using the condition that $(x_1, y_1)$ is on the ellipse we then find

$$p^2 = a^2b^2/(a^2 + b^2 - r^2).$$

But, if $p$ and $r$ are connected by a functional relation $f(p,r) = 0$, then the differential equation satisfied by the roulette is

$$f(y, y[1 + y'^2]^{\frac{1}{2}}) = 0.$$

We thus find the required differential equation to be

$$dy/dx = \pm[y^2(a^2 + b^2) - a^2b^2 - y^4]^{\frac{1}{2}}/y^2.$$

Differentiating again and setting this second derivative equal to zero we find that for a point of inflection on the roulette we must have

$$y^2 = 2a^2b^2/(a^2 + b^2).$$

Substituting this in the first derivative we find, where $\theta$ is the inclination of the first inflectional tangent above the origin,

$$\tan\theta = (a^2 - b^2)/2ab = e^2/2(1 - e^2)^{\frac{1}{2}}.$$

But this angle $\theta$ is also the required angle of the inclined plane, for it is clearly the least angle which causes the roulette to be everywhere a descending curve in the direction of motion down the inclined plane.

It is assumed in the above that the cylinder is carefully placed in its most stable position upon the inclined plane. If this were not so the disturbing feature of momentum would enter into the problem, and the roulette down the inclined plane would not have to be everywhere descending.

Also solved by F. G. Fender, South Orange, N. J.; C. E. Springer, University of Oklahoma; and the proposer.

## QUICKIES

**From time to time as space permits this department will publish problems which may be solved by laborious methods, but which with the proper insight**

**may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.**

**Q 18.** Mr. Precision is an executive who makes a fetish of regularity. He lives in the suburbs, and takes the 4:15 every afternoon. His chauffeur, through long association, has acquired the same regular habits. Thus, he always drives at exactly the same speed; and he arranges to arrive at the station just as the train pulls in. As the car stops, Mr. Precision gets in, and they immediately start for home.

One day, however, Mr. Precision had a headache, and decided to take the 3:05, which arrives at the suburban station one hour earlier than the 4:15. As he left his office, he told his secretary to 'phone his home. It happened that the young lady had a date that evening; her thoughts were far away, and she forgot to make the call.

When Mr. Precision got off at Barrington, his car wasn't there. Figuring that the exercise might do him some good, he started off on foot, walking, as was his custom, at three miles an hour. Presently, he saw the car approaching. He waved to the chauffeur, who turned as Mr. Precision came up, and drove him the rest of the way home. As they drew up at the house, Mr. Precision glanced at his watch, and saw that he was home ten minutes earlier than usual.

What was the speed of the car? [Submitted by J. S. Cromelin].

**Q 19.** Show that $x^4 - 5x^3 - 4x^2 - 7x + 4 = 0$ has no negative roots. [Submitted by R. E. Horton].

**Q 20.** Prove that the medians $AA'$, $BB'$, $CC'$ of a triangle $ABC$ are concurrent.

## ANSWERS

**A 18.** The car saved 5 minutes on the outward journey, so Mr. Precision had been walking 55 minutes. Hence the car's speed was 11 times Mr. Precision's speed or 33 miles per hour.

**A 19.** The equation may be rewritten in the form $5x^3 + 7x = x^4 - 4x^2 + 4$. For every negative $x$, the left-hand member is negative and the right-hand member, $(x^2 - 2)^2$, is positive, so no negative $x$ can satisfy the original equation.

**A 20.** Since the medians bisect the sides, $(AB')(CA')(BC') = (B'C)(A'B)(C'A)$. Therefore, by the converse of Ceva's theorem, the medians are concurrent.

# MATHEMATICAL MISCELLANY

*Edited by*

Charles K. Robbins

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: Charles K. Robbins, Dept. of Mathematics, Purdue University, Lafayette, Indiana.

A professor lecturing to a graduate class said, "This is obvious— or is it?" then went into his office and returned after 30 minutes and continued, "Yes, it is obvious."

" Mr. James:

I wonder if you will please explain what is a puzzle to me.

Wireless World of London some time ago gave a short cut, on slide rule, for the often used Egyptian Triangle diagonal, equal to square root of sum of the squares, or $\sqrt{a^2 + b^2}$ .

The procedure is to add 1 to the square of the ratio and multiply its root by the short leg—The example given for impedance, also used for capacity and resistances, is 28 and 18 = 33.3. Answer is found by:— Divide 28, by 18 on *D* scale, square on *A* scale is 2.42. Add 1, move unity on slide to 3.42, find root on *D* scale (1.85) multiply by 18, answer 33.3 on *D* scale. Only two settings of cursor and two of slide are needed, and no additions involved.

Like everything in Math. Mag. except pi, this is too deep for an 1893 sixth grader—thats me. I cannot figure out how the added 1 above makes it work. Will you please explain it?" .

(*Many can explain but who can furnish the best explanation for a "6th grader" ? Try it, please. — Ed.*)

H. Brocard in *L'Intermédiaire des Mathématiciens,* **11**, 302, (1904) gives E. Lemoine, *Nouvelles Annales* (1870), pp. 311-316 as authority for his statement that Euler proposed the impossible construction of a triangle given the orthocenter, incenter and the centroid.

Can any reader give an earlier date for the appearance of problems of this type?

D. L. MacKay

Manchester Depot

Vermont

## *The Symmedian Point*

By way of passing along pet methods of proof or exposition, according to the invitation in the September issue, I would like to develop a property of the symmedian point synthetically. For readers unfamiliar with the geometry of the triangle, we may define a symmedian as the line symmetric to a median across a bisector of the angle from which the median issues. The three symmedians are concurrent at the symmedian point.

In his brief but stimulating text on College Geometry, P. H. Daus calls attention to a characteristic property of the symmedian point, that the sum of the squares of its distances from the sides of a triangle is a minimum. He then implies that a synthetic, direct proof is not known, calling attention to the methods of calculus or else the use of a certain algebraic identity in establishing this property. Actually, the proof is possible using the following proposition from R. A. Johnson's Modern Geometry: the sum of the squares of the distances of a point from the vertices of a triangle is at a minimum for the centroid. The point for which the sum of the squares of its distances from the sides is a minimum would thus have to be the centroid of the triangle formed by its projections on the sides of the given triangle, and then by the converse of a property given by Daus, i.e. the symmedian point is the centroid of its pedal triangle, it would have to be the symmedian point. The proof of this converse offers no difficulty as the direct proof can be reversed.

## *Special Caustics Treated Synthetically*

Pure geometry may be used to bypass the conventional methods of calculus in the treatment of the caustic, the envelope of light rays reflected from some curve. The two simplest caustics are those formed by parallel rays reflected from the concave side of a semicircle and that in which rays from a point on a circle are reflected by the same circle. Osgood, in his Advanced Calculus, begins the discussion of the former by using a diagram to show that the inclination of a reflected ray is twice the inclination of the radius to the point of reflection, where incoming light is moving in the direction of initial line. This is the beginning of the analytic treatment consisting of formulating the equation of the reflected ray, differentiating with respect to the parametric angle, eliminating this parameter, and identifying the result as an epicycloid of two cusps. However, the reader may show that when a circle rolls on a circle twice as large in generating the epicycloid of two cusps, the tangent always passes through the far end of the diameter of the rolling circle which extends through the center of the fixed circle, and that the inclination of the tangent is twice that of the diameter. This is a characteristic property of the epicycloid of two cusps, and thus Osgood's preliminary examination of the diagram leads immediately to the conclusion without the rather long and ingenious analysis.

In the second type of caustic, the reflected ray can be seen to turn 3/2 times as fast as the radius to the point of reflection. By an examination of a cardioid being generated by a circle rolling on a fixed circle of the same size, it can be seen that the tangent to the cardioid moves and rotates exactly as does the reflected ray of light. Hence we conclude without further analysis that this caustic is the cardioid.

A. S. C. at Flagstaff J. H. Butchart

# OUR CONTRIBUTORS

*J. M. Thomas,* Professor of Mathematics at Duke University, received the Ph.D. from the University of Pennsylvania in 1923. He has been National Research Fellow at Harvard, Paris and Princeton 1924-27, Assistant Professor at Pennsylvania 1927-30, visitor at the Institute for Advanced Study 1936-37, and Managing Editor of the Duke Mathematical Journal 1936-44. His principal field of interest is differential equations. Prof. Thomas is the author of the book "Differential Systems" in the American Mathematical Society Colloquium series.

*John P. Gill,* Assistant Professor of Mathematics and Statistics, Florida State University, received his formal education at Penn State, and the Universities of Alabama (A.B., M.A.), Michigan and Texas (Ph.D. '50). Dr. Gill taught at the Universities of Alabama and Texas and has served as a research statistician with the Federal Reserve Bank of Dallas, and with the War Assets Administration in Houston. He is especially interested in statistics as applied to business and economics.

*Roger Cook Osborn,* Instructor in Applied Mathematics, University of Texas, was born in Texas in 1920. He attended the University of Texas (B.A. '40, M.A. '42) and taught in the Texas public schools before taking his present position in 1942. Since then he has also served for two years as a naval officer. Currently Mr. Osborn is a candidate for the Ph.D. degree in Applied Mathematics at the University of Texas.

*Aristotle D. Michal,* Professor of Mathematics, California Institute of Technology, was born in Smyrna, Asia Minor, in 1899. He attended Clark University (A.B. '20, A.M. '21) and the Rice Institute (Ph.D. '24). After two years as a National Research Fellow at Chicago, Harvard and Princeton, Dr. Michal became Assistant Professor of Mathematics at Ohio State University. He joined the faculty of the California Institute in 1929. Prof. Michal has served on the Council of the American Mathematical Society and also as an Associate Secretary of the Society. Since 1947 he has been an editor of the Mathematics Magazine. Dr. Michal has contributed many research papers to U.S. and foreign mathematical journals on subjects ranging from differential geometry to functional analysis and topological groups and is currently engaged in preparing the manuscript for a book on these subjects. He is also the author of the text, "Matrix and Tensor Calculus with Applications to Elasticity and Aeronautics".

## Outstanding Books

### from McGRAW-HILL

**PLANE TRIGONOMETRY. Alternate Edition**

By E. RICHARD HEINEMAN. 184 pages, (*with tables*) **$3.00**

This edition, available with or without tables, contains 1420 problems nearly all of which differ from those appearing in the regular edition. It offers the instructor an additional selection of problems.

**BUSINESS MATHEMATICS. New 3rd edition**

By CLEON C. RICHTMEYER and JUDSON W. FOUST, Central Michigan College of Education. 441 pages, **$3.75**

Requiring only elementary and secondary school mathematics, this text seeks to acquaint the student with a wide variety of topics. Each topic is developed at length, described in detail, and accompanied by step-by-step illustrations.

**THE DEVELOPMENT OF MATHEMATICS**

By ERIC TEMPLE BELL, Professor of Mathematics, California Institute of Technology. 637 pages, $5.00

Tells with freshness and vigor the absorbing story of the part played by mathematics in the evolution of civilization, from about 4000 B.C. to the present day. The book clearly describes the chief principles, methods, and theories of mathematics which have survived, and presents the major trends.

**ANALYTIC GEOMETRY**

By R.D. DOUGLASS and S.D. ZELDIN, Massachusetts Institute of Technology 216 pages **$3.00**

Simple and direct in approach, and containing a large number of illustrative examples, this text presents the essential topics of elementary analytic geometry, both plane and solid.

**CALCULUS AND ANALYTIC GEOMETRY**

By CECIL THOMAS HOLMES, Bowdoin College. 416 pages, $4.75

Designed for a combination course in which the concepts and techniques of the calculus are the main objectives. Although calculus is emphasized, the essentials of analytic geometry are presented in sufficient detail for those who intend to major in mathematics.

*Send for copies on approval*

**McGRAW-HILL BOOK COMPANY, Inc.**

**330 West 42nd Street** **New York 18, N.Y.**